A Commodity-Money Refinement in Matching Models

Neil Wallace*
Department of Economics
The Pennsylvania State University
608 Kern
University Park, Pa., 16802
neilw@psu.edu;

Tao Zhu
Department of Economics
Cornell University
442 Uris Hall
Ithaca, NY 14853
tz34@cornell.edu

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*Corresponding author: Neil Wallace, Department of Economics, The Pennsylvania State University, 608 Kern, University Park, Pa., 16802; email: neilw@psu.edu; tel 814-863-3805; fax 814-863-4775.

Abstract

We apply a commodity-money refinement to matching models in which people meet in pairs and buyers make take-it-or-leave-it offers to sellers. The refinement is applied by attaching a utility value to nominal money and letting that value approach zero. An equilibrium satisfies the refinement if it is such a limit. We show that the refinement eliminates a class of non full-support steady states.

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1 Introduction

The conception of money as fiat money, an intrinsically useless object, has been used in models for a long time. It was used in the classical-dichotomy model, even though that model was developed and used at a time when actual money was gold or silver. Ignoring the commodity aspect of money was convenient because it produced a simple and strong prediction: allocations are independent of the amount of money. However, if actual money is, in fact, a commodity—albeit a "low value" commodity usable, for example, only as wall-paper or fuel—then we should take seriously only those equilibria which are limits of commodity-money equilibria as the commodity "value" approaches zero. Here, we describe some implications of this refinement (selection device) in some models in which trade occurs between pairs of people and in which the buyer (of goods) makes a take-it-or-leave-it offer to the seller. As we will see, the refinement eliminates a class of non full-support steady states.

We deal with two fairly familiar background models. One is a model in which people are endowed at alternating dates, a special case of Bewley [1]. In our version, at each date each person with an endowment, a seller, meets a random person without an endowment, a buyer. The other model is an indivisible-money version of the random-matching model first set out by Shi [2] and by Trejos and Wright [3], a multi-good model with specialization in consumption and production and no double coincidences. Each model has non full-support monetary steady states that do not satisfy the refinement. In the first model, even though there is no obvious source of heterogeneity, a steady state with the degenerate distributions of money holdings for buyers and sellers that are consistent with competitive trade does not satisfy the refinement. In the second model, the refinement produces a strengthened non-neutrality result.

The idea of applying a commodity-money refinement to fiat-money equilibria is not new. Its consequences in overlapping generations models are well-known. Some such models and others have a class of competitive equilibria in which the value of a fixed stock of fiat money converges to zero. The refinement is known to eliminate those. It also eliminates non-monetary equilibria unless those are the only equilibria.

2 The Commodity "Value" of Money

There are two main ways to model the commodity value of money. One is money as jewelry: wear it to get some utility *and* then keep it or trade it. The other is money as fuel: use it to get some utility *and* lose it. Here, we adopt the fuel specification, although most of the results can be obtained from either specification. In particular, we let the utility payoff from using an amount of money m be given by the function $\epsilon\gamma(m)$, where $\epsilon \geq 0$ is a "productivity-of-money" parameter. We assume that $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, is differentiable, bounded, strictly increasing, and concave, and satisfies $\gamma(0) = 0$ and $\gamma'(0)$ finite.

Our models are discrete-time models with one pairwise meeting per date. Each person enters a date with some money and then meets someone. Each person can decide how to use his money. In general, it is allocated among the amounts traded, held, and used up. All our claims are about steady states in which, by definition, no money is used up. A steady state is defined below for each model studied. A fiat-money ($\epsilon = 0$) steady state satisfies the commodity-money refinement if it is a limit of commodity-money steady states as $\epsilon \to 0$, where an appropriate limit is defined below for each model.

3 Alternating Endowments with Divisible Money

There is one perishable and divisible good per date. There are two equally sized intervals of people, which we label group 1 and group 2. Each member of group 1 has an income stream in the form of goods given by $(\omega, 0, \omega, 0, ...)$, while each member of group 2 has an income stream in the form of goods given by $(0, \omega, 0, \omega, ...)$, where $\omega > 0$. Everyone has the same preferences represented by expected discounted utility with discount factor $\beta \in (0, 1)$ and period utility function $u : \mathbb{R}_+ \to \mathbb{R}$. The function u is strictly increasing, strictly concave, differentiable, and, without loss of generality, $u(0) = 0$. In addition, we assume a lower bound on marginal utility at zero: $\beta u'(0) > 2u'(\omega)$.

There exists a fixed stock of divisible money. At each date a person with endowment 0, the buyer, meets a person with endowment ω, the seller. Each trading partner sees the money of his or her trading partner, but any other information about the partner's trading history is private.[1]

[1] If there is centralized competitive trade at each date that is limited to trading money for the good, then there is a steady state in which every person with endowment 0 starts

A steady state is a pair of value functions defined on the amount of money held and a pair of measures that describe the distribution of money holdings, both of which pertain to the beginning of a date prior to meetings. We let v_0 denote the value function for someone with endowment 0 and v_ω denote that for someone with endowment ω and we let π_0 and π_ω be the respective measures. To describe steady-state conditions, it is helpful to first set out the take-it-or-leave-it choice problem of a buyer with m_0 who meets a seller with m_ω.

Problem 1 *Choose* $(c, z_0, m) \in \mathbb{R}^3_+$ *to maximize* $u(c) + \epsilon\gamma(z_0) + \beta v_\omega(m)$ *subject to* $c \leq \omega$, $z_0 + m \leq m_0$, *and*

$$\begin{aligned} & u(\omega - c) + \max_{z_\omega}[\beta v_0(m_\omega + m_0 - m - z_0 - z_\omega) + \epsilon\gamma(z_\omega)] \\ \geq \ & \max_{z}[u(\omega) + \beta v_0(m_\omega - z) + \epsilon\gamma(z)]. \end{aligned} \tag{1}$$

In this problem, the z variables represent the amounts of money used up. To save space, in (1) we have omitted the obvious non negativity constraints that limit the choices of z_ω and z. Let $g_0(m_0, m_\omega)$ be the maximized objective and let $g_\omega(m_0, m_\omega)$ be the implied value of the left-hand side of (1). Also, let $P(m_0, m_\omega)$ be the set of maximizers for m.

The steady state must satisfy

$$v_0(m_0) = \int g_0(m_0, m_\omega) d\pi_\omega(m_\omega), \ \ v_\omega(m_\omega) = \int g_\omega(m_0, m_\omega) d\pi_0(m_0). \tag{2}$$

Also, let $p(.,.) : \mathbb{R}^2_+ \to \mathbb{R}_+$ be a selection of $P(.,.)$ that is Borel measurable and let the set-valued mappings ζ_0 and ζ_ω on the Borel sets of $\mathbb{R}_+$ be defined by

$$\zeta_0[0, m] = \{(m_0, m_\omega) \in \mathbb{R}^2_+, m_\omega + m_0 - p(m_0, m_\omega) \leq m\}, \tag{3}$$

$$\zeta_\omega[0, m] = \{(m_0, m_\omega) \in \mathbb{R}^2_+, p(m_0, m_\omega) \leq m\}. \tag{4}$$

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the period with the same amount of money and every person with endowment ω starts with no money. As is well-known, the steady state consumptions are given by the solution to $\beta u'(c) = u'(\omega - c)$, where c is the consumption of the person with endowment 0. In this steady state, all money changes hands at each date. It is straightforward to show that this equilibrium satisfies the refinement.

Let π denote the product Borel measure $\pi_0 \times \pi_\omega$ on $\mathbb{R}^2_+$. Then, a steady state must also satisfy

$$1 = \int m d\pi_0(m) + \int m d\pi_\omega(m), \tag{5}$$

$$\pi_0[0,m] = \pi(\zeta_0[0,m]),\ \pi_\omega[0,m] = \pi(\zeta_\omega[0,m]), \tag{6}$$

where, in (5), we have set the mean holding of money to be one half.

That is, we say that $(v_0, v_\omega, \pi_0, \pi_\omega)$ is a steady state if it satisfies (2)-(6), Also, we say that a steady state is a degenerate steady state if $\pi_0(\{1\}) = 1$ and $\pi_\omega(\{0\}) = 1$. And we say that an $\epsilon = 0$ (fiat) steady state is a monetary steady state if trade occurs in some meetings.

3.1 A degenerate fiat monetary steady state

Here we assume that $\epsilon = 0$. With no commodity "value" attached to money, problem 1 simplifies to

Problem 2 *Choose* $(c, m') \in [0, \omega] \times [0, m_0]$ *to maximize* $u(c) + \beta v_\omega(m')$ *subject to*

$$u(\omega - c) + \beta v_0(m_\omega + m_0 - m') \geq u(\omega) + \beta v_0(m_\omega). \tag{7}$$

If (c, m') solves problem 2, then either $c < \omega$ or $c = \omega$. If $c < \omega$, then (7) holds at equality. In this case, $u(\omega) - \beta[v_0(m_\omega + m_0 - m') - v_0(m_\omega)] \geq 0$ and we can solve the constraint at equality for c obtaining

$$c = f[v_0(m_\omega + m_0 - m') - v_0(m_\omega)], \tag{8}$$

where $f : [0, u(\omega)/\beta] \to [0, \omega]$ is defined by

$$f(x) = \omega - u^{-1}[u(\omega) - \beta x]. \tag{9}$$

(Notice that f is strictly increasing and strictly concave with $f(0) = 0$ and $f'(0) = \beta/u'(\omega)$.) If $c = \omega$, then $u(\omega) - \beta[v_0(m_\omega + m_0 - m') - v_0(m_\omega)] \leq 0$. In any case, we conclude that

$$c = \begin{cases} \omega \text{ if } u(\omega) \leq \beta[v_0(m_\omega + m_0 - m') - v_0(m_\omega)] \\ f[v_0(m_\omega + m_0 - m') - v_0(m_\omega)] \text{ otherwise} \end{cases}. \tag{10}$$

In addition, the following notation is useful. Let $c^* \in (0, \omega)$ satisfy

$$u(\omega) = \beta u(c^*) + u(\omega - c^*). \tag{11}$$

(Because $x \mapsto h(x) \equiv \beta u(x) + u(\omega - x)$ is strictly concave on $[0, \omega]$, $h(0) = u(\omega)$, $h(\omega) < u(\omega)$, and $h'(0) = \beta u'(0) - u'(\omega) > 0$, c^* exists and is unique.)

The first proposition gives some properties of a degenerate monetary steady state.

Proposition 1 *If $(v_0, v_\omega, \pi_0, \pi_\omega)$ is a degenerate monetary steady state, then (i) $v_0(0) = \beta v_\omega(0)$; (ii) $v_0(1) - v_0(0) = u(c^*)$; (iii) $v_\omega(0) = u(\omega)/(1 - \beta^2)$; (iv) $v_0(m) = v_0(0)$ for $m \in (0, 1)$; (v) $v_\omega(m) = v_\omega(0)$ for $m \in (0, 1)$.*

Proof. (i) This follows from problem 2.

(ii) By the definition of a degenerate monetary steady state, there must be (c, m') with $c > 0$ and $m' = 0$ that solves problem 2 for $(m_0, m_\omega) = (1, 0)$. It follows that $v_0(1) = u(c) + \beta v_\omega(0)$, and, hence, from part (i) that

$$v_0(1) - v_0(0) = u(c). \tag{12}$$

If $c = \omega$, then $v_0(1) - v_0(0) = u(\omega)$. But by (7), $\beta[v_0(1) - v_0(0)] \geq u(\omega)$, a contradiction. So $c < \omega$ and (7) holds at equality. Then,

$$\beta[v_0(1) - v_0(0)] = u(\omega) - u(\omega - c). \tag{13}$$

It follows from (12) and (13) that $\beta u(c) = u(\omega) - u(\omega - c)$. But then $c > 0$ and the definition of c^* imply $c = c^*$.

(iii) Let (c, m') solve problem 2 for $(m_0, m_\omega) = (1, 0)$. As just established, (7) holds at equality. Hence, $v_\omega(0) = u(\omega) + \beta v_0(0)$. This and part (i) imply the result.

(iv) Let $m \in (0, 1)$. Because a buyer with m can offer m to a seller with 0, $v_0(m) \geq u(c) + \beta v_\omega(0)$, where c is the maximum amount of the good obtained by offering m. Also, $v_0(m) < v_0(1)$; otherwise, a buyer with 1 will not offer 1 to a seller with 0. It follows that $v_0(m) - v_0(0) < v_0(1) - v_0(0)$, and, therefore, that $c = f[v_0(m) - v_0(0)]$. This, $v_0(m) \geq u(c) + \beta v_\omega(0)$, and part (i) imply

$$v_0(m) - v_0(0) \geq u[f(v_0(m) - v_0(0))]. \tag{14}$$

However, by part (ii),

$$v_0(1) - v_0(0) = u[f(v_0(1) - v_0(0))]. \tag{15}$$

Because (14) has the form $x \geq u[f(x)]$ and (15) has the form $x = u[f(x)]$ and because $u'(0)f'(0) = \beta u'(0)/u'(\omega) > 1$, it follows that any solution to (14) satisfies either $v_0(m) = v_0(0)$ or $v_0(m) \geq v_0(1)$. Because the latter has been ruled out, it follows that $v_0(m) = v_0(0)$.

(v) Let $m \in (0,1)$. We have $\beta v_\omega(0) = v_0(0) = v_0(m) \geq \beta v_\omega(m) \geq \beta v_\omega(0)$, where the penultimate inequality follows because the buyer can choose $m' = m$ and the last follows from free disposal. Therefore, $\beta v_\omega(0) = \beta v_\omega(m)$, as required. ■

The next proposition establishes existence of a degenerate monetary steady state.

Proposition 2 *There exists a degenerate monetary steady state.*

Proof. Let π_0^* and π_ω^* be such that $\pi_0^*(\{1\}) = 1$ and $\pi_\omega^*(\{0\}) = 1$. We show that there exist value functions v_0^* and v_ω^* consistent with π_0^* and π_ω^*. The value functions are step functions with jumps at the integers. The proof has two parts: first, we use a fixed point argument to produce the candidates for v_0^* and v_ω^*; then we show that they and π_0^* and π_ω^* constitute a steady state. Because we are constructing step functions with jumps at the integers, the functions are determined by their values at the integers.

In what follows, i denotes an integer. Let $v_0^*(1) - v_0^*(0) \equiv D$, where $v_0^*(0)$ and $v_0^*(1)$ are given by proposition 1, and let $\mathbb{R}_+^\infty$ denote the set of nonnegative sequences. Let

$$\mathbf{V}_0 = \{v_0 \in \mathbb{R}_+^\infty : v_0(0) = v_0^*(0), v_0(1) = v_0^*(1), v_0(i) - v_0(i-1) \in [0, D]\}$$

and let

$$\mathbf{V}_\omega = \{v_\omega \in \mathbb{R}_+^\infty : v_\omega(i) = u(\omega) + \beta v_0(i) \text{ for some } v_0 \in \mathbf{V}_0\}.$$

Let the mapping $\Phi = (\Phi_0, \Phi_\omega) : \mathbf{V}_0 \times \mathbf{V}_\omega \to \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty$ be defined by

$$\begin{aligned} \Phi_0(v_0, v_\omega)(i) \quad &= \max_{(c,m') \in [0,\omega] \times \{0,\ldots,\omega\}} [u(c) + \beta v_\omega(m')] \\ &\quad \text{subject to } \beta[v_0(i - m') - v_0(0)] \geq u(\omega) - u(\omega - c), \end{aligned} \tag{16}$$

and by

$$\Phi_\omega(v_0, v_\omega)(i) = u(\omega) + \beta \Phi_0(v_0, v_\omega)(i).$$

Let $\mathbb{R}_+^\infty$ be equipped with the topology of pointwise convergence. Then $\mathbf{V}_0 \times \mathbf{V}_\omega$ is compact. Also, $\mathbf{V}_0 \times \mathbf{V}_\omega$ is convex. And by the Theorem of Maximum, Φ_0 is continuous. It follows that Φ_ω is continuous. If $\Phi(v_0, v_\omega) \in \mathbf{V}_0 \times \mathbf{V}_\omega$, then it follows immediately that Φ has a fixed point. To show that $\Phi(v_0, v_\omega) \in \mathbf{V}_0 \times \mathbf{V}_\omega$, it suffices to show that $\Phi_0(v_0, v_\omega) \in \mathbf{V}_0$.

It is clear that $\Phi_0(v_0, v_\omega)(0) = v_0^*(0)$. Now consider $\Phi_0(v_0, v_\omega)(1)$. The choice $m' = 0$ implies a payoff of $u(c^*) + \beta v_\omega(0)$, while $m' = 1$ implies $\beta v_\omega(1)$. By proposition 1, the former is strictly better than the latter. Therefore, $\Phi_0(v_0, v_\omega)(1) = v_0^*(1)$. It remains to show that $\Phi_0(v_0, v_\omega)(i) - \Phi_0(v_0, v_\omega)(i-1) = \delta \in [0, D]$ for $i \geq 2$. It is clear that $\delta \geq 0$. Now let $\Phi_0(v_0, v_\omega)(i) = u(c) + \beta v_\omega(m')$. Either $m' = 0$ or $m' > 0$. If $m' > 0$, then a lower bound on $\Phi_0(v_0, v_\omega)(i-1)$ is obtained by evaluating the objective in (16) at $(c, m'-1)$. This gives $\Phi_0(v_0, v_\omega)(i-1) \geq u(c) + \beta v_\omega(m'-1)$. Then $\delta \leq \beta[v_\omega(m') - v_\omega(m'-1)] \leq \beta^2 D$. If $m' = 0$, there are two possible cases.

Case 1: $u(\omega) \leq \beta[v_0(i) - v_0(0)]$. Then $c = \omega$, and a lower bound on $\Phi_0(v_0, v_\omega)(i-1)$ can be obtained by evaluating the objective in (16) at $(c', 0)$ for some c'. If $u(\omega) \leq \beta[v_0(i-1) - v_0(0)]$, then $c' = \omega$, and $\delta \leq 0$. Otherwise, $u(\omega) - u(\omega - c') = \beta[v_0(i-1) - v_0(0)]$. This and $u(\omega) \leq \beta[v_0(i) - v_0(0)]$ imply $u(\omega - c') \leq \beta[v_0(i) - v_0(i-1)]$. It follows that $\delta \leq u(\omega) - u(c') < u(\omega - c') < D$.

Case 2: $u(\omega) > \beta[v_0(i) - v_0(0)]$. Then $c = f[v_0(i) - v_0(0)]$, and a lower bound on $\Phi_0(v_0, v_\omega)(i-1)$ can be obtained by evaluating the objective in (16) at $(f[v_0(i-1) - v_0(0)], 0)$. It follows that

$$\begin{aligned} \delta &\leq u[f(v_0(i) - v_0(0))] - u[f(v_0(i-1) - v_0(0))] \\ &\leq u[f(v_0(i) - v_0(i-1))] \leq u[f(D)] = D, \end{aligned}$$

where the second inequality follows from concavity of u and f.

A fixed point of Φ gives values of v_0^* and v_ω^* at the integers, and, therefore, determines the step functions. To complete the proof, we show that the step functions v_0^* and v_ω^* and π_0^* and π_ω^* constitute a steady state. First, we claim that

$$v_0^*(m) = \max_{(c,m') \in [0,\omega] \times [0,m]} [u(c) + \beta v_\omega^*(m')] \tag{17}$$

subject to

$$\beta[v_0^*(m - m') - v_0^*(0)] \geq u(\omega) - u(\omega - c)\}.$$

Because v_0^* and v_ω^* are step functions, the constraint $m' \in [0, m]$ in (17) can be replaced by $m - m' \in \{0, 1, ..., int(m)\}$, where $int(m)$ is the largest integer no greater than m. Then, for $m = i$, the claim holds because v_0^* and v_ω^* are derived from a fixed point of Φ. For $m \in [i, i+1)$, the claim follows from the result for integers and the fact that v_0^* and v_ω^* are step functions. Next, we show that in problem 2 with $v_0 = v_0^*$, $v_\omega = v_\omega^*$, and $(m_0, m_\omega) = (1, m)$, the implied seller's payoff is $v_\omega^*(m)$. For $m = i$, let (c, m') be a solution of the problem. Because $[v_0^*(i+1) - v_0^*(i)] \leq D < \omega/\beta$, (7) is binding. Therefore, $v_\omega^*(i) = u(\omega) + \beta v_0^*(i)$. For $m \in [i, i+1)$, this follows again from the result for integers and the fact that v_0^* and v_ω^* are step functions. ■

Thus, as one might expect, there is a degenerate steady state for the above model. However, because the value functions are not concave, the steady state may not survive lotteries in meetings. If lotteries are allowed, then the constructed steady state survives under an additional restriction: the buyer must want to offer the amount 1 with probability 1, rather than with some lower probability. That is the case if $\frac{u'(c^*)}{u'(\omega - c^*)} \geq \beta$ (if c^* is no greater than the competitive equilibrium consumption in footnote 1). That restriction may or may not hold. It does not hold if β is close enough to 1 because $c^* \to \omega$ as $\beta \to 1$. It holds, for example, if $u(c) = \sqrt{c}$ and $\beta^2 \leq 1/3$.

3.2 Application of the refinement

Now we apply the commodity-money refinement. For this model, we say that an $\epsilon = 0$ steady state $(v_0, v_\omega, \pi_0, \pi_\omega)$ is the limit of commodity-money steady states $(v_{0\epsilon}, v_{\omega\epsilon}, \pi_{0\epsilon}, \pi_{\omega\epsilon})$ as $\epsilon \to 0$ if (i) $(\pi_{0\epsilon}, \pi_{\omega\epsilon})$ converges weakly to (π_0, π_ω) and (ii) $(v_{0\epsilon}, v_{\omega\epsilon})$ converges uniformly to (v_0, v_ω) over $[0, 1]$ and converges pointwise over $(1, \infty)$.[2] We say that an $\epsilon = 0$ steady state satisfies the refinement if it is such a limit. The next proposition shows that no $\epsilon = 0$ degenerate monetary steady state satisfies the refinement.

Proposition 3 *If $(v_0, v_\omega, \pi_0, \pi_\omega)$ is an $\epsilon = 0$ monetary steady state that is degenerate, then $(v_0, v_\omega, \pi_0, \pi_\omega)$ does not satisfy the commodity-money refinement.*

[2] We suspect that pointwise convergence of value functions is sufficient, but our proof uses the stronger notion.

Proof. Assume by contradiction that $(v_0, v_\omega, \pi_0, \pi_\omega)$ satisfies the refinement and let $\{(v_{0\epsilon}, v_{\omega\epsilon}, \pi_{0\epsilon}, \pi_{\omega\epsilon})\}$ be the commodity-money steady states whose limit is $(v_0, v_\omega, \pi_0, \pi_\omega)$.

First, we claim that for sufficiently small ϵ, $\pi_{\omega\epsilon}(0) > 0.5$. To see why, notice that for sufficiently small ϵ, $\pi_{\omega\epsilon}[0, 0.1]$ is sufficiently close to 1. Now if $\pi_{\omega\epsilon}(0) \leq 0.5$, then there exists $l \in (0, 0.1)$ with $\pi_{\omega\epsilon}[l, 0.1]$ sufficiently close to 0.5. But now consider $v_{0\epsilon}(1-l)$. It is feasible for a buyer with $1-l$ to offer $1-m$ to a seller with $m \in [l, 0.1]$. By convergence, $v_{0\epsilon}(1)$ is close to $v_0(1)$ and $v_{0\epsilon}(1-l)$ is close to $v_0(1-l)$, and by proposition 1, $v_0(1-l) = v_0(0)$. But then, such an offer implies that $v_{0\epsilon}(1-l)$ is not close to $v_0(1-l)$, a contradiction.

Now, let d be the unique positive solution to the equation $x = 0.5u[f(x)]$. Notice that the properties of f and the lower bound on $u'(0)$ imply that d exists and that $d < D \equiv v_0(1) - v_0(0)$. Now fix $m \in (0, 1)$ and let ϵ be such that $(v_{0\epsilon}, v_{\omega\epsilon}, \pi_{0\epsilon}, \pi_{\omega\epsilon})$ satisfies $\pi_{\omega\epsilon}(0) > 0.5$ and $v_{0\epsilon}(m) - v_{0\epsilon}(0) < d$. A lower bound on $v_{0\epsilon}(m)$ is obtained by considering the following two feasible actions for a buyer with m: use up all of m or offer m to a seller with zero. The second action induces $f[v_{0\epsilon}(m) - v_{0\epsilon}(0)]$ amount of the good from the seller because $v_{0\epsilon}(m) - v_{0\epsilon}(0) < d < D < u(\omega)/\beta$. Then those two actions for the buyer give us the following inequality:

$$v_{0\epsilon}(m) - \beta v_{\omega\epsilon}(0) \geq \max\{\epsilon\gamma(m), 0.5u[f(v_{0\epsilon}(m) - v_{0\epsilon}(0))]\}. \qquad (18)$$

Let $x \equiv v_{0\epsilon}(m) - v_{0\epsilon}(0)$. By $\beta v_{\omega\epsilon}(0) = v_{0\epsilon}(0)$, we can rewrite (18) as

$$x \geq \max\{\epsilon\gamma(m), 0.5u[f(x)]\}. \qquad (19)$$

But (19) implies $x \geq d$, a contradiction. ■

3.3 A full-support steady state

With two added technical assumptions, a bound on $u'(0)$ and a bound on individual holdings of money, it can be shown that the model with $\epsilon = 0$ has a full-support steady state with a strictly increasing and strictly concave value function. Zhu [6] used those assumptions in a closely related model to establish existence of such a steady state, and the arguments, although lengthy, can be adapted to the model of this section. To show that such a steady state satisfies the refinement, we proceed as follows.

As a sequence of steady states, we take the constant sequence, each term of which is the above $\epsilon = 0$ full-support steady state on $[0, B]$, where B is the bound on individual holdings. The results in [6] also imply that the steady state value functions have positive left derivatives at B.[3] We have to show that this $\epsilon = 0$ steady state is a steady state for all sufficiently small ϵ. That is, we have to show that if ϵ is small, then no money is used up, or that for all $\omega \geq x$ and $s \in \{0, \omega\}$,

$$\beta[v_s(\omega) - v_s(\omega - x)] \geq \epsilon\gamma(x). \tag{20}$$

But

$$\beta[v_s(\omega) - v_s(\omega - x)] \geq \beta v_s'(B)x \geq \epsilon\gamma'(0)x \geq \epsilon\gamma(x), \tag{21}$$

where $v_s'(B)$ is the left derivative of v_s at B and where the first inequality follows from concavity of v_s, the second holds for $\epsilon \leq \beta v_s'(B)/\gamma'(0)$, and the last follows from concavity of γ. Therefore, (20) holds for sufficiently small ϵ.

Notice that the bound B plays a crucial role in showing that no one uses up money. If we had adopted the money-as-jewelry specification, then preservation of the stock of money would not arise as an issue. However, even for that specification, there is no existence proof for B infinite.

4 Random Matching with Indivisible Money

Time is discrete. There is a $[0, 1]$ continuum of each of $N \geq 3$ types of infinitely lived people, and there are N distinct produced and perishable types of divisible goods at each date. A type n person, $n \in \{1, 2, ..., N\}$, produces only good n and consumes only good $n+1$ (modulo N). Each person maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. For a type n person, utility in a period is $u(q_{n+1}) - q_n$, where q_{n+1} is the amount of good $n+1$ consumed and q_n is the amount of good n produced. The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, continuously differentiable and satisfies $u(0) = 0$, $u'(\infty) = 0$, and $u'(0) = \infty$.

There exists a fixed stock of money. Let the average money holding per type be denoted by $\overline{m}$, the (smallest) unit of money by $\Delta(> 0)$, and the exogenous (finite) upper bound on individual money holdings by B. As in Zhu [5], there are lower bounds on $\overline{m}/\Delta$ and $B/\overline{m}$, and B/Δ is an integer. Let $B_\Delta = \{0, \Delta, 2\Delta, ..., B\}$ denote the set of possible individual holdings of money.

[3] A proof is available upon request.

In each period, people are randomly matched in pairs. A meeting between a type n person, a seller, and a type $n+1$ person, a buyer, is called a single-coincidence meeting. Other meetings are not relevant. In meetings, as above, types and money holdings are observable, but any other information about a person's trading history is private.

A symmetric steady state is a value function, v, and a measure, π, both defined on B_Δ that pertain to the start of a date prior to meetings. The choice problem of a buyer with money m_b who meets a seller with money m_s is identical to that described by problem 1 except that choices are limited to B_Δ and imply holdings in B_Δ.[4] We let $g_b(m_b, m_s)$ and $g_s(m_b, m_s)$ be the implied payoffs to the buyer and seller, respectively, and let $P(m_b, m_s)$ be the set of maximizers for the buyer's post-trade amount of money. Then the value function satisfies

$$v(m) = [N - (N-2)\beta]^{-1} \sum_{m'} \pi(m')[g_b(m, m') + g_s(m', m)]. \tag{22}$$

We allow all possible randomizations over the elements of $P(m_b, m_s)$. Let $\Lambda(m_b, m_s)$, a set of measures on B_Δ, be given by

$$\Lambda(m_b, m_s) = \{\lambda(.; m_b, m_s) : \lambda(m; m_b, m_s) = 0 \text{ if } m \notin P(m_b, m_s)\}. \tag{23}$$

Then π satisfies

$$1 = \sum \pi(m)m, \tag{24}$$

and

$$\pi(m) = \frac{1}{2} \textstyle\sum_{m_b, m_s} \pi(m_b)\pi(m_s)[\lambda(m; m_b, m_s)$$

$$+ \lambda(m_b + m_s - m; m_b, m_s)] \tag{25}$$

for some $\lambda(.; m_b, m_s) \in \Lambda(m_b, m_s)$. (As a convention, $\lambda(z) = 0$ if $z \notin B_\Delta$ in (25).) Then, we say that (v, π) is a steady state if it satisfies (22)-(25).

[4] As this suggests, we are here requiring that offers be deterministic. Lotteries are discussed below.

4.1 Fiat monetary steady states

For $\epsilon = 0$, Zhu [5] shows that there exists a steady state with v strictly increasing and strictly concave, with $v(0) = 0$, and with π having full support. Let us call any such steady state a *nice* steady state. Because the value function is strictly increasing and concave, any nice steady state satisfies the refinement.[5]

Existence of such nice steady states implies existence of non full-support steady states. We first describe them and then show that none satisfy the refinement. The model has three exogenous nominal quantities, $(\Delta, \overline{m}, B) \equiv z$. Let k be an integer that exceeds unity and consider $z' = kz$ and $z'' = (\Delta, k\overline{m}, kB)$. As might be expected, neutrality holds for the comparison between z and for z', neutrality in the sense of real allocations. What about the comparison between z and z''? In terms of real allocations, z'' has strictly more steady states. In particular, no nice steady state for z'' has the same real allocation as any steady state for z. However, any steady state for z has an equivalent (in terms of real allocations) steady state for z''. An equivalent steady state is produced by replacing the trade of m units of money under z by a trade of km units under z'' and by letting (v'', π''), the equivalent steady state for z'', be given as follows: for each $m \in B_\Delta$, let $\pi''(km) = \pi(m)$, and let $v''(km) = v(m)$ and $v''(km + x) = v''(km)$ for $x = \Delta, \Delta + 1, ..., k\Delta - \Delta$. That is, v'' is a step function with jumps at km. All of these claims are proved in Zhu [5].[6]

This comparison between steady states for z and for z'' raises the following question. Under z'', are we likely to see a nice steady state implying non-neutrality relative to z or might we see a steady state which is equivalent to one for z? We answer that by applying the refinement.

4.2 Application of the refinement

As above, an $\epsilon = 0$ steady state satisfies the refinement if there exists a sequence of commodity-money steady states that converges to it. Here, because

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[5] The argument is a simpler version of that given for the model of alternating endowments.

[6] As noted above, Zhu [5] proves existence of a nice steady state for a version without lotteries. However, his arguments can be applied almost without change to a version with lotteries. And such existence implies existence of non full-support steady states via the same construction used in the text. In contrast to the model with divisible money, lotteries do not impose any additional restrictions for such existence.

steady states are finite dimensional, convergence is defined in the usual way. We show that the refinement rules out steady-state value functions which are not strictly increasing. We prove this in two steps.

The first proposition shows that if (v, π) is an $\epsilon = 0$ steady state and v is not strictly increasing, then v is not strictly increasing at the lower end of the support of π.

Proposition 4 *Let* (v, π) *be an* $\epsilon = 0$ *steady state. Let* $a = \min\{m : \pi(m) > 0\}$. *(i)* $v(a) = 0$. *(ii) If* $a > 0$, *then* $v(a + \Delta) = 0$. *(iii) If* v *is not strictly increasing, then* $v(\Delta) = 0$.

Proof. (i) If $v(a) > 0$, then a buyer with a has to trade with someone. That gives rise to an inflow into holdings less than a, a contradiction.

(ii) Assume by contradiction that $v(a + \Delta) > 0$. Because a buyer with a can offer Δ to a seller with a, it follows that $v(a) > 0$, which contradicts (i).

(iii) Assume by contradiction that $v(\Delta) > 0$. Let $m = \min\{m' : v(m') = v(m' + \Delta)\}$. It follows that $m > 0$ and $v(m) > 0$. Hence, the buyer with m must trade with some probability. That is, there is a positive probability that the buyer with m makes an offer $p \geq \Delta$ to some sellers. A buyer with $m + \Delta$ has the same probability of meeting those sellers and can emulate the offers made by the buyer with m. If so, then the buyer with $m + \Delta$ ends up with $m + \Delta - p$ and the buyer with m ends up with $m - p$ with positive probability. But the definition of m and $p \geq \Delta$ imply $v(m + \Delta - p) > v(m - p)$. That, in turn, implies $v(m + \Delta) > v(m)$, a contradiction. ■

Now we apply the refinement.

Proposition 5 *If* (v, π) *is an* $\epsilon = 0$ *monetary steady state with* v *not strictly increasing, then* (v, π) *does not satisfy the commodity-money refinement.*

Proof. Assume by contradiction that (v, π) satisfies the refinement. As in proposition 4, let $a = \min\{m : \pi(m) > 0\}$. Let d denote the unique positive solution to the equation $x = \frac{\pi(a)}{2N} u(\beta x)$. By the definition of the refinement, there exists a steady state $(v_\epsilon, \pi_\epsilon)$ with v_ϵ close to v and

$$\sum_{x<a} \pi_\epsilon(a) < \frac{1}{2N} [\frac{\pi(a)}{2}]^2, \tag{26}$$

and

$$\pi_\epsilon(a) > \frac{\pi(a)}{2}. \tag{27}$$

Notice that (26) must hold because by the definition of a, $\sum_{x<a} \pi(x) = 0$. (The proof is written as if $a > 0$. As noted below, $a = 0$ is a simple special case that does not make use of (26).) Now, in the $(v_\epsilon, \pi_\epsilon)$ steady state, a buyer with a either trades with a seller with a or not. If a trades with a, then (27) implies that the inflow into holdings less than a is bounded below by $\frac{1}{N}[\frac{\pi(a)}{2}]^2$. But then $\sum_{x<a} \pi_\epsilon(x) > \frac{1}{2N}[\frac{\pi(a)}{2}]^2$, which contradicts (26). Therefore, a does not trade with a. (If $a = 0$, this is the only possibility.) And because $(v_\epsilon, \pi_\epsilon)$ is a steady state, the person with a does not use up any money.

Now, consider someone with $a + \Delta$. Among the feasible actions for this person are two: use up Δ and offer Δ when meeting a seller with a. Because someone with a does not use up any money and does not trade with a seller with a, those feasible actions imply the following inequality:

$$v_\epsilon(a + \Delta) - v_\epsilon(a) \geq \max\{\epsilon\gamma(\Delta), \frac{\pi(a)}{2N} u[\beta v_\epsilon(a + \Delta) - \beta v_\epsilon(a)]\}.$$

But this implies $v_\epsilon(a + \Delta) - v_\epsilon(a) \geq d$. However, because proposition 4 implies that $v(a + \Delta) - v(a) = 0$, the last inequality violates the condition that v_ϵ is close to v. ■

We have not proved that any steady state that satisfies the refinement has full support. We have shown that a non full-support steady state that satisfies the refinement cannot have a step-function value function. And we know that it cannot have a concave value function because Zhu [5] shows that concavity of a steady state value function implies that the accompanying π has full support. We have not ruled out a non full-support steady state with a strictly increasing value function that is not concave.

5 Concluding Remarks

Our results show that the commodity-money refinement has bite in some models of pairwise trade. As in most other models, the equilibria which do not satisfy the refinement are those in which the marginal value of money is zero. An exception is Zhou [4], in which an equilibrium with a step-function value function satisfies the refinement.[7] In any case, if monetary objects do, indeed, have some value as commodities, then the refinement should always be applied.

[7] In Zhou's model, the good is available in one indivisible unit, money is divisible, holdings of money are private information, and sellers post a price of one unit of the good.

There are, of course, other refinements that might be considered for steady states of the models we have studied. An obvious one, which perhaps should not even be called a refinement, is local stability. For now, we have no results for other refinements or for local stability.

Throughout, we have assumed take-it-or-leave-it offers by buyers because existence of a full-support steady state with a strictly increasing and concave value function has been established only for that bargaining outcome.[8] That being the case, it seems premature to discuss a refinement for versions with other bargaining outcomes.

References

[1] T. Bewley, The optimum quantity of money, in *Models of Monetary Economies*, (J. Kareken and N. Wallace, Eds.) pp. 169-210, Federal Reserve Bank of Minneapolis, Minneapolis, 1980.

[2] S. Shi, Money and prices: a model of search and bargaining, *J. of Econ. Theory* 67 (1995) 467-498.

[3] A. Trejos, and R. Wright, Search, bargaining, money and prices, *J. of Political Economy* 103 (1995) 118-141.

[4] R. Zhou, Does commodity money eliminate the indeterminacy of equilibria?, *J. of Econ. Theory* 110 (2003) 176-190.

[5] T. Zhu, Existence of a monetary steady state in a matching model: indivisible money, forthcoming, *J. of Econ. Theory.*

[6] T. Zhu, Existence of a monetary steady state in a matching model: divisible money, miemo, Cornell University, 2003.

[8]The mapping studied to establish existence preserves concavity of value functions under take-it-or-leave-it offers, but not under other bargaining outcomes. That is why Zhu's [5] approach does not apply directly under other bargaining outcomes.