Existence of a Monetary Steady State in a Matching Model: Indivisible Money*

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Abstract

Existence of a monetary steady state is established for a random matching model with divisible goods, indivisible money, an arbitrary bound on individual money holdings, and take-it-or-leave-it offers by consumers. The background environment is that in papers by Shi and by Trejos and Wright. The monetary steady state shown to exist has nice properties: the value function, defined on money holdings, is strictly increasing and strictly concave, and the measure over money holdings has full support. *Journal of Economic Literature* Classification Number: E40.

Key Words: matching model; indivisible money; steady states; existence theorem.

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1 Introduction

Shi [8] and Trejos and Wright [11] introduce a matching model of money with divisible goods. While the model builds on the indivisible goods model of Kiyotaki and Wright [5], the introduction of divisible goods permits output and prices to be determined as part of an equilibrium. Trejos and Wright show that equilibrium under a bargaining rule is easily formulated for general individual money holdings. However, existence of a monetary equilibrium has been established only for special versions. Here, I give a general existence proof for indivisible money. In particular, under the bargaining rule that potential consumers make take-it-or-leave-it offers, I prove that there exists a steady state with a value function defined on money holdings that is strictly increasing and strictly concave and with a measure over money holdings that has full support. The only assumptions are an arbitrary bound on individual money holdings and lower bounds on (a) the marginal utility of consumption at zero and (b) the ratio of the average stock of money to the size of the smallest unit of money.

Proving existence is difficult because the general model has endogenous heterogeneity of money holdings. Most researchers simplify or avoid the endogeneity of the distribution of money holdings by making special assumptions.[1] One exception is Molico [7]. He studies the model numerically and claims to find monetary steady states for divisible money and unbounded individual holdings. My results—and those in a companion paper on divisible money—provide a basis for interpreting his numerical results. Another exception is Taber and Wallace [10], who study the same model with commodity money, money with a direct utility payoff. They establish existence of a steady state with a concave and strictly increasing value function. I extend their result to fiat money. To deal with fiat money, I show that there exists a steady state

[1] Green and Zhou [3] and Zhou [13] assume indivisible goods and divisible money. Green and Zhou [4] assume divisible goods and divisible money, but make preference assumptions that effectively make goods indivisible. Camera and Corbae [2] consider the same model as I study. For a special region of the parameter space, one which makes discounting disappear as the bound on individual holdings grows, they construct a steady state in which one unit of money is offered in every trade. In all these cases, which involve a trade of either one unit of the good or one unit of money, the steady state distribution is shown to be geometric or truncated geometric. No such special distribution is implied generally. Cavalcanti [1] assumes a unit bound on money holdings and a large number of kinds of money. Shi [9] and Lagos and Wright [6] make special assumptions that produce a degenerate distribution of money holdings.

for the corresponding commodity money version in which the value of money is bounded away from zero as the direct utility payoff approaches zero.[2]

The properties of the steady state shown to exist—monotonicity and concavity of the value function and full support of the measure—are important.[3] It is well known that monetary steady states with step-function value functions and non-full-support measures can be easily constructed. One implication of the full-support property is that the steady state that I show exists is different in terms of allocations from those steady states. Moreover, as I show later, that difference gives rise to a non-neutrality result.

2 The Model

As noted above, the model is essentially that in [8, 11].

2.1 Environment

Time is discrete, dated as $t \geq 0$. There is a $[0,1]$ continuum of each of $N \geq 3$ types of infinitely lived agents, and there are N distinct produced and perishable types of divisible goods at each date. A type n agent, $n \in \{1, 2, ..., N\}$, produces only good n and consumes only good $n+1$ (modulo N). Each agent maximizes expected discounted utility with discount factor $\beta \in (0,1)$. For a type n agent, utility in a period is $u(q_{n+1}) - q_n$, where $q_{n+1} \in \mathbb{R}_+$ is the amount of good $n+1$ consumed and $q_n \in \mathbb{R}_+$ is the amount of good n produced. The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, and continuously differentiable and satisfies $u(0) = 0$ and $u'(\infty) = 0$. In addition, there is a lower bound on $u'(0)$ which is specified later.

There exists a fixed stock of money which is perfectly durable. Money is symmetrically distributed across the N specialization types. Let the average money holding be denoted by $\overline{m}$, the (smallest) unit of money by $\Delta (> 0)$, and the exogenous (finite) upper bound on individual money holdings by B. I assume that Δ is small relative to $\overline{m}$ and B is large relative to $\overline{m}$ with lower bounds on $\overline{m}/\Delta$ and $B/\overline{m}$ specified later. ($B > \overline{m}$ is necessary for trade to occur.) Also, B/Δ is assumed to be an integer. Let $B_\Delta = \{0, \Delta, ..., B\}$ denote the set of possible individual holdings of money.

[2] The approach used here is likely to be applicable to models in which the source of heterogeneity in money holdings is preference shocks rather than random meetings.

[3] Taber and Wallace [10] did not describe the support.

In each period, agents are randomly matched in pairs. A meeting between a type n agent and a type $n+1$ agent is called a single-coincidence meeting. Other meetings are not relevant. In meetings, agents' types and money holdings are observable, but any other information about an agent's trading history is private.

2.2 Definition of Equilibrium

In single-coincidence meetings, the potential consumer makes a take-it-or-leave-it offer, (p, q), where p is the amount of money offered and q is the amount of production demanded. Let $w_t(x)$ be the expected discounted value of holding x amount of money at the start of period t, prior to date t matching, where $w_t : B_\Delta \to \mathbb{R}_+$ is nondecreasing. Consider a date t single-coincidence meeting between a consumer with x amount of money and a producer with m amount of money. Let

$$\Gamma(x, m) = \{p \in B_\Delta : p \leq \min\{x, B - m\}\}, \tag{1}$$

the set of feasible offers of money. Assuming, as is standard, that the producer accepts all offers which leave him no worse off, an optimal offer satisfies $p \in \Gamma(x, m)$ and $q = \beta w_{t+1}(m+p) - \beta w_{t+1}(m)$, where the equality for q says that the lower bound on the producer's gain-from-trade, zero, is attained. Therefore, the consumer's problem reduces to $\max_{p\in\Gamma(x,m)}\{u[\beta w_{t+1}(m+p) - \beta w_{t+1}(m)] + \beta w_{t+1}(x-p)\}$. To express this objective function more succinctly, it is convenient to introduce a symbol for an increment in a function: for any function $k : B_\Delta \to \mathbb{R}$, let $k(x, y) \equiv k(x) - k(x - y)$. Using this shorthand and dropping the time subscript on the value function, for a nondecreasing $w : B_\Delta \to \mathbb{R}_+$ and $(x, m) \in B_\Delta^2$, let

$$f(x, m, w) = \max_{p\in\Gamma(x,m)} \{u[\beta w(m + p, p)] + \beta w(x - p)\}, \tag{2}$$

and

$$p(x, m, w) = \arg\max_{p\in\Gamma(x,m)} \{u[\beta w(m + p, p)] + \beta w(x - p)\}. \tag{3}$$

That is, when w is the value of money at the start of the next period, $f(x, m, w)$ is the payoff for a consumer with x who meets a producer with m while $p(x, m, w)$ is the set of optimal offers of money.

Because $p(x,m,w)$ is discrete, and, may, therefore, be multi-valued, it is important for existence to allow all possible randomizations over the elements of $p(x,m,w)$. In order to describe the law of motion for the distribution of money holdings, it is convenient to express randomizations over the post-trade money holdings of consumers. Therefore, I define the set of randomizations, a set of probability measures on B_Δ, as

$$\Lambda(y,m,w) = \{\lambda(.;y,m,w) : \lambda(x;y,m,w) = 0 \text{ if } x \notin y - p(y,m,w)\}, \tag{4}$$

where $\lambda(x;y,m,w)$ is the probability that consumers with y (pre-trade) in meetings with producers with m (pre-trade) offer $y-x$ (and, therefore, end up with x).

Let $\pi_t(x)$ denote the fraction of agents holding x amount of money at the start of period t, so that π_t is a measure on B_Δ. The law of motion for π_{t+1} can be expressed as

$$\pi_{t+1}(x) = \frac{N-2}{N}\pi_t(x) + \frac{1}{N}\sum_{y,m}\pi_t(y)\pi_t(m)[\lambda(x;y,m,w_{t+1}) + \lambda(m+y-x;m,y,w_{t+1})] \tag{5}$$

for some

$$\lambda(.;y,m,w_{t+1}) \in \Lambda(y,m,w_{t+1}). \tag{6}$$

Note that $\lambda(m+y-x;m,y,w_{t+1})$ is the probability that producers with y (pre-trade) in meetings with consumers with m (pre-trade) end up with x. The value function, $w_t(x)$, satisfies

$$w_t(x) = \frac{N-1}{N}\beta w_{t+1}(x) + \frac{1}{N}\sum_m \pi_t(m) f(x,m,w_{t+1}). \tag{7}$$

This follows from the fact that the payoff to being a producer with x is $\beta w_{t+1}(x)$.

I can now state the relevant definitions.

Definition 1 *Given π_0, a sequence $\{w_t, \pi_{t+1}\}_{t=0}^{\infty}$ is an equilibrium if it satisfies (1) – (7). A monetary equilibrium is an equilibrium with positive consumption and production. A pair (w,π) is a steady state if $\{w_t, \pi_{t+1}\}_{t=0}^{\infty}$ with $w_t = w$ and $\pi_{t+1} = \pi$ for all t is an equilibrium for $\pi_0 = \pi$.*

3 Existence of a Monetary Steady State

To establish the existence of a monetary steady state, the following assumptions are maintained from now on.

(A1)$u'(0) > [2/(R\beta)]^2$, where $R \equiv [N-(N-1)\beta]^{-1}$.
(A2)$B/\overline{m} \geq 4$.
(A3)$\overline{m}/\Delta \geq \beta W/D$, where D is the unique solution of $u'(D) = [2/(R\beta)]^2$ and W is the unique solution of $N(1-\beta)W = u(\beta W) + N$.

In this model, existence always requires a lower bound on $u'(0)$ because a producer has to see a future reward from producing. (Note that $R < 1$. Also, if (w, π) is a steady state, then (7) can be written as $w(x) = R\sum_m \pi(m) f(x, m, w)$, an expression that is used repeatedly below.) Assumptions (A2) and (A3) say that the set of individual holdings is large enough (relative to the average holding). (Note that $D < \beta W$ because $[2/(R\beta)]^2 > N(1-\beta)/\beta$. Also, as will be shown in Lemma 1, $W - 1$ can be taken to be an upper bound on steady state value functions.)

Proposition 1 *Under assumptions* $(A1) - (A3)$*, there exists a steady state* (w, π) *where* w *is strictly increasing and strictly concave and* π *has full support.*

The rest of this section is devoted to a proof of Proposition 1. The proof consists of 10 lemmas. For the proof, it is convenient to normalize the nominal objects—$\overline{m}$, Δ, and B—by setting $\overline{m} = 1$. Moreover, it is convenient, althought not essential, to assume $\overline{m} \in B_\Delta$.

I start by defining the main correspondences used. These are essentially implied by (1)-(7). Let $\mathbf{W}$ be the set of concave and nondecreasing functions from B_Δ to $[0, W]$.[4] Let $\mathbf{\Pi}$ be the set of measures on B_Δ satisfying the unit mean condition.

Let the single-valued mapping Φ_w on $\mathbf{W} \times \mathbf{\Pi}$ be defined by

$$\Phi_w(w, \pi)(x) = \frac{N-1}{N}\beta w(x) + \frac{1}{N}\sum\nolimits_m \pi(m) f(x, m, w). \tag{8}$$

Let the correspondence Φ_π on $\mathbf{W} \times \mathbf{\Pi}$ be defined by

[4] A real function w defined on B_Δ is concave if $2w(x) \geq w(x-\Delta) + w(x+\Delta)$ for $0 < x < B$.

$$\Phi_\pi(w,\pi) = \{\nu : \nu(x) = \frac{N-2}{N}\pi(x) + \frac{1}{N}\sum_{y,m}\pi(y)\pi(m)[\lambda(x;y,m,w) \quad (9)$$
$$+ \lambda(m+y-x;m,y,w)] \text{ for some } \lambda(.;y,m,w) \in \Lambda(y,m,w)\}.$$

Finally, let $\Phi = (\Phi_w, \Phi_\pi)$.

Results in the next two lemmas are essentially from [10].

Lemma 1 *(i)*$\mathbf{W} \times \mathbf{\Pi}$ *is convex and compact. (ii)*$\Phi(w,\pi) \subset \mathbf{W} \times \mathbf{\Pi}$ *with* $\Phi_w(w,\pi)$ *bounded above by* $W-1$*. (iii)*Φ *is convex-valued, compact-valued, and upper hemicontinuous (u.h.c.).*

Proof. For the bound in part (ii), note that $N\Phi_w(w,\pi)(x) \leq (N-1)\beta w(x) + u[\beta w(x)] + \beta w(x) \leq N\beta W + u(\beta W) = NW - N$, where the first inequality follows from (8) and the equality from the definition of W. Other results can be found in the proof of [10, Proposition 1]. ■

Next, I introduce a perturbation of the correspondence Φ, which can be interpreted as assigning some direct utility to money. Let the real function h on B_Δ be defined by

$$h(x) = x/4 \text{ if } x \leq 4, \quad h(x) = 1 \text{ if } x > 4.$$

Let $\mathbf{K}$ be the set of concave and nondecreasing functions from B_Δ to $[0, W-1]$. (Note that $\mathbf{K} \subset \mathbf{W}$.) For a positive integer n, let the correspondence $\Phi_n = (\Phi_{w,n}, \Phi_{\pi,n})$ on $\mathbf{K} \times \mathbf{\Pi}$ be defined by

$$\Phi_n(w,\pi) = \Phi(w + h/n, \pi). \quad (10)$$

Lemma 2 *The mapping* Φ_n *has a fixed point.*

Proof. Because $w \in \mathbf{K}$ implies $w + h/n \in \mathbf{W}$ and $(w,\pi) \mapsto (w + h/n, \pi)$ is continuous, by Lemma 1, Φ_n is convex-valued, compact-valued, and u.h.c. with $\Phi_n(w,\pi) \subset \mathbf{K} \times \mathbf{\Pi}$. Because $\mathbf{K} \times \mathbf{\Pi}$ is convex and compact, it follows from Kakutani's Fixed Point Theorem that Φ_n has a fixed point. ■

The next lemma, the main ingredient in the existence proof, establishes a uniform (with respect to n in (10)) lower bound on the value functions of the fixed points of Φ_n.

Lemma 3 *If* (w, π) *is a fixed point of* Φ_n*, then* $w(4) \geq D/\beta - 1/n$.

Proof. Assume by contradiction that $w(4) < D/\beta - 1/n$. Let $w + h/n$ be denoted by φ. The proof is split into two steps. In the first step, we calculate a lower bound on $f(4, m, \varphi) - f(4 - \Delta, m, \varphi)$ for $m \leq 2$. In the second step, we draw contradictions based on this bound. In this and subsequent proofs, we suppress the dependence of f and p on φ or w. Also, from now on, for an interval I and a measure μ on B_Δ, we denote $\mu(I \cap B_\Delta)$ by μI.

Step 1. To get the lower bound, consider two possibilities for $p(4-\Delta, m)$ for each $m \leq 2$, according to whether an element of $p(4-\Delta, m)$ does or does not exceed 2. First, assume $p(4 - \Delta, m) \ni p \geq 2$. Because a consumer with money holding 4 can make the same offer as a consumer with $4 - \Delta$ does, and, hence, get the same amount of the consumption good, it follows that

$$f(4, m) - f(4 - \Delta, m) \geq \beta\varphi(4 - p, \Delta) \geq \beta\varphi(2, \Delta) > \beta w(2, \Delta),$$

where the second inequality follows from concavity of φ. Next, assume $p(4 - \Delta, m) \ni p < 2$. Because $m \leq 2$, we have $p + \Delta + m \leq 4 \leq B$. Hence the consumer with 4 can offer $p + \Delta$ to producers with m and end up with the same money holding as the consumer with $4 - \Delta$. It follows that

$$\begin{aligned} & f(4, m) - f(4 - \Delta, m) \\ & \geq u[\beta\varphi(m + p + \Delta, p + \Delta)] - u[\beta\varphi(m + p, p)] \\ & > u'[\beta\varphi(m + p + \Delta, p + \Delta)]\beta\varphi(m + p + \Delta, \Delta) \\ & > u'(D)\beta\varphi(m + p + \Delta, \Delta) \\ & \geq u'(D)\beta\varphi(4, \Delta) \\ & > u'(D)\beta w(4, \Delta), \end{aligned} \tag{11}$$

where the second inequality follows from the Mean Value Theorem and strict concavity of u, the third from $\beta\varphi(m+p+\Delta, p+\Delta) < \beta\varphi(4) = \beta[w(4)+1/n] < D$ and strict concavity of u, and the fourth from concavity of φ. Let $l = \min\{\beta w(2, \Delta),\, u'(D)\beta w(4, \Delta)\}$. Then for $m \leq 2$, $f(4, m) - f(4 - \Delta, m) > l$.

Step 2. Because (w, π) is a fixed point of Φ_n, we have

$$w(x) = R(N - 1)\beta h(x)/n + R \textstyle\sum_m \pi(m) f(x, m).$$

Therefore,

$$w(x, \Delta) = R(N - 1)\beta h(x, \Delta)/n + R \textstyle\sum_m \pi(m)[f(x, m) - f(x - \Delta, m)].$$

Because $f(x,m) \geq f(x-\Delta,m)$ for all m, it follows that for $m^* \in B_\Delta$,

$$w(x,\Delta) \geq R\sum_{m=0}^{m^*}\pi(m)[f(x,m)-f(x-\Delta,m)]. \tag{12}$$

Because the average holding is 1, $\pi[0,2] \geq 1/2$. Then by (12) and Step 1,

$$w(4,\Delta) > Rl/2. \tag{13}$$

Now consider the two possible values of l. If $l = u'(D)\beta w(4,\Delta)$, then by (13),

$$w(4,\Delta) > Rl/2 = (R\beta/2)u'(D)w(4,\Delta) = [2/(R\beta)]w(4,\Delta) > w(4,\Delta),$$

a contradiction. So it must be that $l = \beta w(2,\Delta)$. Then by (13),

$$w(4,\Delta) > Rl/2 = (R\beta/2)w(2,\Delta). \tag{14}$$

To rule this out, we calculate a lower bound on $f(2,m) - f(2-\Delta,m)$ for $m \leq 2$. Let $p \in p(2-\Delta,m)$. Because $p \leq 2-\Delta$ and $m \leq 2$, we have $p+\Delta+m \leq 4 \leq B$. Hence a consumer with 2 can offer $p+\Delta$ to producers with m. It follows that

$$\begin{aligned}
&f(2,m) - f(2-\Delta,m)\\
&\geq u[\beta\varphi(m+p+\Delta,p+\Delta)] - u[\beta\varphi(m+p,p)]\\
&> u'(D)\beta w(4,\Delta)\\
&> u'(D)\beta(R\beta/2)w(2,\Delta),
\end{aligned}$$

where the second inequality follows exactly the logic used in (11) and the last from (14). Let $l' = u'(D)\beta(R\beta/2)w(2,\Delta)$. By (12) and $\pi[0,2] \geq 1/2$, we have

$$w(2,\Delta) > Rl'/2 = (R\beta/2)^2u'(D)w(2,\Delta) = w(2,\Delta),$$

a contradiction.[5] ■

In some respects, the ingredients in the proof of Lemma 3 have analogues in the simple case with $B = \Delta = 1$. In my proof, I require that there be

[5] In this proof, Δ is simply required to be no greater than unity, the average holding. If the average holding $\overline{m}$ is not unity, then we require $\Delta \leq \overline{m}$ and we redefine h as $h(x) = x/(4\overline{m})$ for $x \leq 4\overline{m}$ and $h(x) = 1$ for $x > 4\overline{m}$. It follows that $w(4\overline{m})$ is bounded below by $D/\beta - 1/n$.

a set of "poor" agents with positive measure. This set plays the role of the agents with 0 when $B = \Delta = 1$. In the proof, the "poor" set is $[0, 2] \cap B_\Delta$ because there is an obvious lower bound on the measure of this set — namely, 1/2. However, other sets would also work. The agents with 4 are like those with holdings of 1 when $B = \Delta = 1$. Of course, when $B = \Delta = 1$, the monetary steady state can be computed directly because the distribution of money holdings and the offers in trades are fixed. The argument here is complicated because very little is known either about the distribution or the offers that agents make.

Now I show that there is a monetary steady state by taking a limit as the direct utility payoff of money approaches zero.

Lemma 4 *Let $\{(w_n, \pi_n)\}$ be a sequence such that (w_n, π_n) is a fixed point of Φ_n. (i)$\{(w_n, \pi_n)\}$ has at least one limit (accumulation) point, denoted (w, π). (ii)(w, π) is a fixed point of Φ. (iii)$w(0) = 0$ and $w(4) \geq D/\beta$.*

Proof. Because $\mathbf{W} \times \mathbf{\Pi}$ is compact, there is a subsequence of $\{(w_n, \pi_n)\}$ that converges to some $(w, \pi) \in \mathbf{W} \times \mathbf{\Pi}$. To simplify the notation, let $\{(w_n, \pi_n)\}$ represent the subsequence converging to (w, π). Because (w_n, π_n) is a fixed point of Φ_n, it follows from (10) that $(w_n, \pi_n) \in \Phi(w_n + h/n, \pi_n)$. Because $(w_n, \pi_n) \to (w, \pi)$, it follows that $(w_n + h/n, \pi_n) \to (w, \pi)$. Because Φ is compact-valued and u.h.c., $(w_n + h/n, \pi_n) \to (w, \pi)$ and $(w_n, \pi_n) \in \Phi(w_n + h/n, \pi_n)$ imply that there is a subsequence of $\{(w_n, \pi_n)\}$ converging to an element of $\Phi(w, \pi)$. Because $\{(w_n, \pi_n)\}$ itself converges to (w, π), it follows that $(w, \pi) \in \Phi(w, \pi)$. Part (iii) is obvious. ■

Any Lemma 4 limit point (w, π) is a monetary steady state according to Definition 1. The remaining lemmas establish properties of any such limit point. In them, (w, π) denotes a Lemma 4 limit point. The next lemma establishes strict monotonicity of w.

Lemma 5 *w is concave and strictly increasing.*

Proof. Concavity is obvious. Assume by contradiction that w is not strictly increasing. By concavity of w, there exists $a > 0$ such that $w(x) = w(a)$ if $x \geq a$ and $w(x) < w(y)$ if $x < y \leq a$. (That is, by concavity, the flat portion of w must occur over a set of the form $\{a, a + \Delta, ...\}$.) It follows that $w(a) > 0$. Hence, there must be a positive probability that a consumer with a makes an offer $p \geq \Delta$ to some producers. A consumer with $a + \Delta$

has the same probability of meeting those producers and can also make the offer p. If so, he ends up with $a + \Delta - p$ and the consumer with a ends up with $a - p$. But then $p \geq \Delta$ implies $w(a + \Delta - p) > w(a - p)$. This, in turn, implies $w(a + \Delta) > w(a)$, a contradiction. ■

Now I turn to establishing that the steady state measure has full support. The full support result relies on some facts about the optimal offers of money, $p(x, m, w)$, and their dependence on x and m.

Lemma 6 *(i)If $p_1 \in p(x, m, w)$ and $p_2 \in p(x + \Delta, m, w)$, then $p_2 - p_1 \in \{0, \Delta\}$. (The consumer's marginal propensity to spend on a given producer is between 0 and 1.)*

(ii)If $p_1 \in p(m, x, w)$ and $p_2 \in p(m, x + \Delta, w)$, then $p_1 \leq p_2 + \Delta$. (For a given consumer, the producer's post-trade money holding is nondecreasing in his pre-trade holding.)

(iii)If $x_2 \geq x_1$ and $m_2 < m_1$, then $\max p(x_2, m_2, w) \leq \max\{x_2 - x_1, m_1 - m_2\} + \min p(x_1, m_1, w)$. (If the consumer is richer and the producer is poorer, then the change in spending is bounded above by the maximum of differences in the consumer's and the producer's holdings.)

(iv)Assume $\min p(x_1, m_1, w) = 0$ and $m_2 \geq m_1$. If $x_2 > x_1$, then $\max p(x_2, m_2, w) \leq x_2 - x_1$. If $x_2 = x_1$, then $\min p(x_2, m_2, w) = 0$. (If a consumer and a producer do not trade, then a richer consumer who meets a richer producer offers at most the consumer's increment.)

(v)If $x > m$, then $0 \notin p(x, m, w)$. (If the consumer is richer than the producer, then there is trade.)

Proof. See the Appendix. ■

In Lemma 6, only part (v) depends on (w, π) being a Lemma 4 limit point. Parts (i)-(iv) depend only on w being concave and strictly increasing. Also, the previous lemmas do not use all of assumption (A3) (see footnote 5). The next lemma does.

Lemma 7 *$\pi(B) > 0$.*

Proof. Assume by contradiction that $x = \max\{m : \pi(m) > 0\} < B$. Because w is concave and bounded above by $W - 1$ and $w(0) = 0$, it follows that $w(x + \Delta, \Delta) < \Delta W/x$. Because the average holding is 1, it follows that

$x \geq 1$. Then by assumption (A3), $w(x+\Delta, \Delta) < D/\beta$. By the definition of x, $0 \in p(x, x)$. It follows that

$$\beta w(x, \Delta) \geq u[\beta w(x+\Delta, \Delta)] > u'(D)\beta w(x+\Delta, \Delta). \tag{15}$$

Also, because $0 \in p(x, x)$ and $\Delta \in \Gamma(x+\Delta, x)$, it follows that

$$f(x+\Delta, x) - f(x, x) \geq u[\beta w(x+\Delta, \Delta)] > u'(D)\beta w(x+\Delta, \Delta). \tag{16}$$

Now, either $\pi[0, x) \geq 1/2$ or $\pi(x) \geq 1/2$. If the latter, then

$$\begin{aligned} w(x+\Delta, \Delta) &= R \textstyle\sum_m \pi(m)[f(x+\Delta, m) - f(x, m)] \\ &> R\pi(x)u'(D)\beta w(x+\Delta, \Delta) \\ &> w(x+\Delta, \Delta), \end{aligned} \tag{17}$$

a contradiction. (Here, the first inequality follows from (16) and $f(x+\Delta, m) > f(x, m)$ for all m.) So $\pi[0, x) \geq 1/2$. Let $m < x$. By Lemma 6 (v), $\min p(x, m) > 0$. Because $p(x, m) \subset \Gamma(x+\Delta, m)$, it follows that $f(x+\Delta, m) - f(x, m) \geq \beta w(x, \Delta)$. Then by the logic used in (17), we have $w(x+\Delta, \Delta) > R\pi[0, x)\beta w(x, \Delta) > (R/2)u'(D)\beta w(x+\Delta, \Delta) > w(x+\Delta, \Delta)$, a contradiction. (Here, the second inequality follows from (15).) ■

In what follows, let *supp* π denote the support of π. The next lemma shows that *supp* π is periodic.

Lemma 8 *Let* $a = \min\{x > 0 : \pi(x) > 0\}$. *Then* B/a *is an integer and* *supp* $\pi = \{0, a, 2a, ..., B\}$.

Proof. The following proof is written as if $a > \Delta$. It also applies if $a = \Delta$, which is a simple special case. In this proof, we let $i, j \in \mathbb{Z}_+$. By Lemma 6 (v), $\min p(a, 0) \geq \Delta$. Now let $n = \max\{i : \min p(a, ia) \geq \Delta\}$.

Claim 1 : $n \geq 1$. Assume by contradiction that $n = 0$. By Lemma 6 (iv), this implies $0 \in p(a, m)$ for $m \geq a$. Then $w(a) > 0$ implies $\pi(0) > 0$. Therefore, by the definition of a, $a \in p(a, 0)$ occurs with probability 1. (Otherwise, $p \in p(a, 0)$ with $p \in (0, a)$ occurs with positive probability, but then $\pi(a)\pi(0) > 0$ implies $\pi(a-p) > 0$.) Let $\rho = \pi(0)$. We have

$$w(a) = R\rho u[\beta w(a)] + R(1-\rho)\beta w(a). \tag{18}$$

We also have $w(2a) \geq R\rho\{u[\beta w(a)] + \beta w(a)\} + R(1-\rho)\beta w(2a)$. Comparing this with (18), we have

$$w(2a, a) \geq R\rho\beta w(a) + R(1-\rho)\beta w(2a, a). \qquad (19)$$

Now let $c = [1 - R(1-\rho)\beta]/(R\rho)$. (Note that $c > 1$.) By (18) and (19), we have $cw(a) = u[\beta w(a)]$ and $w(2a, a) \geq \beta w(a)/c$. Let $g = \beta w(a)/c$. It follows that $u[\beta w(2a, a)] \geq u(\beta g) > cg = \beta w(a)$. (Here, the second inequality follows from $g < w(a)$ and $cw(a) = u[\beta w(a)]$.) But $n = 0$ implies $\beta w(a) \geq u[\beta w(2a, a)]$, a contradiction.

Claim 2 : $\pi(ja) > 0$ *for* $j = 0, 1, ..., n+1$. We proceed by induction: for $j = 1, ..., n$, $\pi(ja) > 0$ implies $\pi(0) > 0$ and $\pi(ja + a) > 0$. By the definition of a, we only need to establish the induction step. By Lemma 6 (iv) and the definition of n, $\min p(a, ja) \geq \Delta$. Then by the definition of a, $a \in p(a, ja)$ occurs with probability 1. It follows that $\pi(0) > 0$ and $\pi(ja + a) > 0$.

Claim 3 : $\pi(x) = 0$ *for* $x \neq ia$ *if* $x \leq na + a$. Suppose otherwise. We first establish the following induction argument: if $n \geq 2$, then for $j = 2, ..., n$, $x \in (ja - a, ja)$ with $\pi(x) > 0$ implies $\pi(x + a) > 0$. To see this, assume that x and j satisfy the conditions. Then by Lemma 6 (iv) and the definition of n, $\min p(a, x) \geq \Delta$. It follows that $a \in p(a, x)$ occurs with probability 1. Hence $\pi(x + a) > 0$. By the contradicting assumption and the induction argument, $\exists\ x \in (na, na + a)$ with $\pi(x) > 0$. Because $a \in p(a, na)$, by Lemma 6 (ii), $\min p(a, x) \geq na + a - x$. Because $0 \in p(a, na + a)$, by Lemma 6 (ii), $\max p(a, x) \leq na + a - x$. Hence $p(a, x) = \{na + a - x\}$. But then $\pi(x - na) > 0$, a contradiction.

If $B = na + a$, then the proof is complete. If not, then we turn to the next two claims.

Claim 4 : $\pi(x) = 0$ *for* $x \neq ia$ *if* $B \geq x > na + a$. We proceed by induction: for $j \geq 1$, $\pi(x) = 0$ for $x \neq ia$ if $x \leq na + ja$ implies $\pi(x) = 0$ for $x \neq ia$ if $x \leq na + ja + a$. By Claim 3, the hypothesis holds for $j = 1$. So it suffices to establish the induction step. Assume by contradiction that $\pi(x) = 0$ for some $x \in (na+ja, na+ja+a)$. By Lemma 6 (v), $\min p(x, 0) > 0$. By Lemma 6 (iii), $\max p(x, 0) \leq na+ja+\min p(a, na+ja) = na+ja$. Because x is not a multiple of a, any feasible value of $p(x, 0)$ makes $\pi(y) > 0$ for some $y \leq na + ja$ where $y \neq ia$, a contradiction.

It follows from Claim 4 and Lemma 7 that B/a is an integer.

Claim 5 : $\pi(na+ja) > 0$ *for* $B/a - n \geq j > 1$. We proceed by induction: for $j \geq 1$, $\pi(na + ja) > 0$ implies $\pi(na + ja + a) > 0$. By Claim 2, the hypothesis holds for $j = 1$. So it suffices to establish the induction step. Let $k = \min\{i : \min p(ia, na + ja) \geq \Delta\}$. (By Lemma 6 (v), k exists.) First assume $k > n + j$ and assume by contradiction that $\pi(na + ja + a) = 0$. Let $l = \min\{i : \pi(ia) > 0, i \geq n + j + 2\}$. By Lemma 6 (iii), $\max p(la, na + ja - a) \leq (l - n - j)a + \min p(na + ja, na + ja) = (l - n - j)a$, where the equality follows from $n + j < k$. By Lemma 6 (v), $\min p(la, na + ja - a) > 0$. But then by Claims 3 and 4, any feasible value of $p(la, na + ja - a)$ makes $\pi(ia)$ > 0 for some $n + j < i < l$, a contradiction. Next assume $k \leq n + j$. By the definition of k, $0 \in p(ka - a, na + ja)$. By Lemma 6 (i) and Claims 3 and 4, this implies that $a \in p(ka, na + ja)$ occurs with probability 1. Then by the induction assumption, $\pi(na + ja + a) > 0$. ■

Now I can prove that π has full support. The proof is by contradiction. If a (see Lemma 8) exceeds Δ, then I can construct a mapping that is concave and strictly increasing and has more than one positive fixed point. However, this mapping can have at most one positive fixed point.[6]

Lemma 9 *supp* $\pi = B_\Delta$.

Proof. By Lemma 8, it suffices to prove that $a = \Delta$. So assume by contradiction that $a \geq 2\Delta$. In this proof, we let $i \in \{1, 2, ..., B/a\}$ and $j \in \{0, 1, ..., B/a\}$.

First, we introduce some notation. Let $\pi(ja)$ be denoted by π_j and $w(ja)$ by w_j. Also, let

$$k_i = w_i - w_{i-1} \text{ and } h_i = w(ia - \Delta) - w_{i-1}.$$

(Note that if $a = \Delta$, then $h_i = 0$.) Let $k = (k_1, k_2, ..., k_{B/a})$ and $h = (h_1, h_2, ..., h_{B/a})$. Let $\alpha = \min_i(h_i/k_i)$. By monotonicity of w, $\alpha \in (0, 1)$. Note that $h \geq \alpha k$ with $h_i = \alpha k_i$ for some i. Let $f(ia, ja)$ be denoted by $f_{i,j}$. Now consider $p(ia, ja)$. By Lemma 6 (i), if $p_1, p_2 \in p(x, m)$, then $|p_2 - p_1| \in \{0, \Delta\}$. Because $a \geq 2\Delta$, this implies that there is at most one element of $p(ia, ja)$ that is equal to na for some $n \in \mathbb{Z}_+$. By Lemma 8, any element of $p(ia, ja)$ that occurs with positive probability is equal to na for some $n \in \mathbb{Z}_+$. Hence, there exists a unique element of $p(ia, ja)$ that is equal

[6] The proof that the mapping has at most one positive fixed point resembles the proof of Zeidler [12, Corollary 7.45 (i), p. 309].

to na for some $n \in \mathbb{Z}_+$ and occurs with probability 1. Let this element be denoted by $\overline{p}(i,j)a$. Finally, let

$$A_{i0} = \{j : \overline{p}(i,j) = \overline{p}(i-1,j)\} \text{ and } A_{i1} = \{j : \overline{p}(i,j) = \overline{p}(i-1,j)+1\}.$$

By Lemma 6 (i), $A_{i0} \cup A_{i1} = \{0, 1, ..., B/a\}$. (Also, note that $A_{i0} \cap A_{i1}$ is empty.)

Next, for each pair of (i,j) and $x \in \{k, h, \alpha k\}$, we define $\phi_{i,j}(x)$ and $\sigma_{i,j}(x)$ according to whether $j \in A_{i0}$ or $j \in A_{i1}$. If $j \in A_{i0}$, then let

$$\phi_{i,j}(x) = \beta x_{i-\overline{p}(i,j)}.$$

Note that $j \in A_{i0}$ implies

$$\phi_{i,j}(k) = f_{i,j} - f_{i-1,j}. \tag{20}$$

Also, by Lemma 6 (i), $j \in A_{i0}$ implies $p(ia - \Delta, ja) = \{\overline{p}(i,j)a\}$. Hence

$$\phi_{i,j}(h) = f(ia - \Delta, ja) - f_{i-1,j}. \tag{21}$$

If $j \in A_{i1}$, then let

$$\sigma_{i,j}(x) = u[\beta(x_{j+\overline{p}(i,j)} + w_{j+\overline{p}(i,j)-1} - w_j)] - u[\beta(w_{j+\overline{p}(i,j)-1} - w_j)].$$

Note that $j \in A_{i1}$ implies

$$\sigma_{i,j}(k) = f_{i,j} - f_{i-1,j}. \tag{22}$$

Also, by Lemma 6 (i), $j \in A_{i1}$ implies $p(ia - \Delta, ja) = \{\overline{p}(i,j)a - \Delta\}$. Hence

$$\sigma_{i,j}(h) = f(ia - \Delta, ja) - f_{i-1,j}. \tag{23}$$

Next, for each i and $x \in \{k, h, \alpha k\}$, let $\theta_i(x)$ be defined by

$$\theta_i(x) = \frac{N-1}{N}\beta x_i + \frac{1}{N}[\textstyle\sum_{j \in A_{i0}} \pi_j \phi_{i,j}(x) + \sum_{j \in A_{i1}} \pi_j \sigma_{i,j}(x)].$$

By (20) and (22), we have

$$\theta_i(k) = k_i. \tag{24}$$

By (21) and (23), we have

$$\theta_i(h) = h_i. \tag{25}$$

(Hence, $\theta = (\theta_1, \theta_2, ..., \theta_{B/a})$, as a mapping from $\{k, h, \alpha k\}$ to $\mathbb{R}_{B/a}$, has multiple positive fixed points.)

Note that $\phi_{i,j}(\alpha k) = \alpha\phi_{i,j}(k)$. Also, by strict concavity of u and $\alpha \in (0,1)$, $\sigma_{i,j}(\alpha k) > \alpha\sigma_{i,j}(k)$. Therefore, $\theta_i(\alpha k) \geq \alpha\theta_i(k)$, and, strictly if A_{i1} is nonempty. We claim that *if* $h_i = \alpha k_i$, *then* A_{i1} *is empty.* Assume by contradiction that $h_i = \alpha k_i$ and A_{i1} is nonempty. It follows that $\theta_i(\alpha k) > \alpha\theta_i(k)$. But then

$$h_i = \theta_i(h) \geq \theta_i(\alpha k) > \alpha\theta_i(k) = \alpha k_i,$$

a contradiction. (Here, the first equality follows from (25) and the second from (24). The first inequality follows from $h \geq \alpha k$.)

Let $n = \min\{i : h_i = \alpha k_i\}$. By the claim, A_{n1} is empty. Because (w, π) is a steady state, A_{11} is nonempty. So $n > 1$. Because A_{n1} is empty, it follows that

$$\begin{aligned}
h_n - \alpha k_n &= \theta_n(h) - \alpha\theta_n(k) \\
&= \frac{N-1}{N}\beta(h_n - \alpha k_n) + \frac{1}{N}\textstyle\sum_j \pi_j[\phi_{n,j}(h) - \alpha\phi_{n,j}(k)] \\
&= \frac{1}{N}\textstyle\sum_j \pi_j\beta[h_{n-\overline{p}(n,j)} - \alpha k_{n-\overline{p}(n,j)}] \\
&\geq \frac{1}{N}\pi_0\beta[h_{n-\overline{p}(n,0)} - \alpha k_{n-\overline{p}(n,0)}].
\end{aligned}$$

By Lemma 6 (v), $\overline{p}(n,0) > 0$. By $0 \in A_{n0}$, $\overline{p}(n,0) < n$. But then $h_i > \alpha k_i$ for $1 \leq i < n$ implies $h_n > \alpha k_n$, a contradiction. ■

Full support allows us to establish strict concavity of the value function.

Lemma 10 *w is strictly concave.*

Proof. See the Appendix. ■

The conclusions of Lemmas 4, 5, 9, and 10 constitute a proof of Proposition 1.

4 Full-support and Non Full-support Steady States

It is well known that monetary steady states with value functions that are step functions can be easily constructed. For example, it is easy to construct a steady state with support $\{0, B\}$ which is identical to one with a unit upper bound on individual holdings ($\Delta = B = 1$). Therefore, it is important to know whether or not a Proposition 1 steady state is different from those steady states in terms of allocations. Moreover, we might like to know in what sense neutrality does or does not hold in the model. To proceed, I begin with a definition of equivalence between steady states that permits us to distinguish between real and nominal differences among steady states. In what follows, for a steady state (w, π), let

$$\pi_*(x) = \begin{array}{c} 0 \text{ if } \exists y < x \text{ with } w(y) = w(x), \\ \pi\{z : z \geq x \text{ with } w(z) = w(x)\} \text{ otherwise.} \end{array}$$

Note that $\pi_* \equiv \pi$ if w is strictly increasing. Otherwise, π_* bunches all quantities with the same payoff at the smallest quantity with that payoff.

Definition 2 *Let (w, π) and (w', π') be steady states. We say that (w', π') is equivalent to (w, π) if there exists a bijection γ from supp π_* to supp π'_* such that if $x \in$ supp π_*, then $w(x) = w'(\gamma(x))$ and $\pi_*(x) = \pi'_*(\gamma(x))$. Let $e \equiv (\overline{m}, \Delta, B)$, the vector of exogenous nominal objects, and let $S(e)$ denote the set of* ***all*** *steady states associated with e. We say that $S(e) \subset S(e')$ if $(w, \pi) \in S(e)$ implies that there exists $(w', \pi') \in S(e')$ with (w', π') equivalent to (w, π). We say that $S(e)$ and $S(e')$ are equivalent if $S(e) \subset S(e')$ and $S(e') \subset S(e)$.*

That is, two steady states are equivalent if the measures of agents assigned to any given expected utility are the same. Exactly in this sense, two equivalent steady states imply the same allocation. And, of course, two non-equivalent steady states imply different allocations. (It is clear from the definition that equivalence between steady states is symmetric and transitive.) If $(w, \pi) \in S(e)$ is a Proposition 1 steady state and $(w', \pi') \in S(e)$ is such that w' is a step function, then they are not equivalent.

As is not surprising, the model implies neutrality for proportional changes in all exogenous nominal quantities. That is, if two economies differ only in their vectors of exogenous nominal objects, e and e', and $e = \alpha e'$ for some $\alpha \in \mathbb{R}_+$, then $S(e)$ and $S(e')$ are equivalent.[7] This is neutrality. However,

[7]Let $e' = (\overline{m}, \Delta, B)$ and $e = \alpha e'$. Let $(w', \pi') \in S(e')$. Under the vector e, let (w, π) be defined as follows. For $n \geq 0$, let $w(n\alpha\Delta) = w'(n\Delta)$ and $\pi(n\alpha\Delta) = \pi'(n\Delta)$. Notice from (3) that $p \in p(x, m, w')$ implies $\alpha p \in p(\alpha x, \alpha m, w)$. Hence $(w, \pi) \in S(e)$. It is clear that (w, π) is equivalent to (w', π').

equivalence does not hold if e and e' differ but are not proportional to each other. Without loss of generality, let $e = (\overline{m}, \Delta, B)$ and $e' = (\overline{m}', \Delta, B')$ with either $\overline{m} > \overline{m}'$ or $B > B'$. In each case, there is no steady state in $S(e')$ equivalent to a Proposition 1 steady state in $S(e)$.[8] Moreover, such non-neutrality is not due solely to the bound. If $e = (k\overline{m}, \Delta, kB)$ and $e' = (\overline{m}, \Delta, B)$ for some integer $k \geq 2$, then $S(e) \supset S(e')$ strictly.[9] Neutrality would hold for this comparison if money were divisible. A surmise is that for any integer $k \geq 2$, some $s \in S(k\overline{m}, \Delta, kB)$ has more trade and higher average welfare than any $s \in S(\overline{m}, \Delta, B)$, but that remains to be established.

5 Concluding Remarks

The model I have studied contains a number of restrictive assumptions. Some matter and some do not. Even though money is indivisible, I have assumed that agents do not trade lotteries. However, it is straightforward to show that Proposition 1 also holds for a version of the model in which offers are lotteries. I have also assumed that individual holdings are bounded. Because, as implied by Proposition 1, the bound is binding, it would be nice to drop it. However, in that case, it is hard to find a sensible way to keep the set of measures compact. I have also assumed the special bargaining outcome associated with consumers making take-it-or-leave-it offers. This assures that the mapping studied preserves concavity of value functions. For other bargaining outcomes, that mapping does not preserve concavity. For these reasons, extensions to unbounded holdings and other bargaining outcomes require different proof techniques.

Appendix

Proof of Lemma 6

Proof. (i)See [10, Lemma 2, p. 967].

[8] The case with $B > B'$ is clear. So consider $\overline{m} > \overline{m}'$. Let $(w, \pi) \in S(e)$ be a Proposition 1 steady state. Assume that $(w', \pi') \in S(e')$ is equivalent to (w, π). Let $supp\ \pi'_* = \{a_0, a_1, ...\}$ with $a_i < a_j$ for $i < j$. Let γ be the bijection in Definition 2. Note that γ is strictly increasing and that $supp\ \pi_* = supp\ \pi$. Then $supp\ \pi = B_\Delta$ implies $\gamma(i\Delta) = a_i$. Hence $\pi(i\Delta) = \pi'_*(a_i)$. But because $i\Delta \leq a_i$, this implies $\overline{m} \leq \overline{m}'$, a contradiction.

[9] Let $(w', \pi') \in S(e')$. Under the vector e, let (w, π) be defined as follows. For $n \geq 0$ and $1 \leq j \leq k-1$, let $w(nk\Delta) = w(nk\Delta + j\Delta) = w'(n\Delta)$, $\pi(nk\Delta) = \pi'(n\Delta)$, and $\pi(nk\Delta + j\Delta) = 0$. It is clear that $(w, \pi) \in S(e)$ and is equivalent to (w', π').

(ii)Assume by contradiction that $p_1 > p_2 + \Delta$. Let $a_1 = \beta w(x + p_2 + \Delta, p_2)$, $a_2 = \beta w(x + p_2 + 2\Delta, p_2 + \Delta)$, $b_1 = \beta w(x + p_1 - \Delta, p_1 - \Delta)$, and $b_2 = \beta w(x + p_1, p_1)$. Because $a_2 - a_1 = \beta w(x + p_2 + 2\Delta, \Delta) > 0$ and $b_2 - b_1 = \beta w(x + p_1, \Delta) > 0$, by the definitions of p_1 and p_2, we have

$$\frac{u(a_2) - u(a_1)}{a_2 - a_1} w(x + p_2 + 2\Delta, \Delta) \leq w(m - p_2, \Delta), \tag{26}$$

$$\frac{u(b_2) - u(b_1)}{b_2 - b_1} w(x + p_1, \Delta) \geq w(m - p_1 + \Delta, \Delta). \tag{27}$$

By the definitions of a_i and b_i, we have $b_1 - a_1 = \beta[w(x+p_1-\Delta) - w(x+p_2+\Delta) + w(x+\Delta, \Delta)]$ and $b_2 - a_2 = \beta[w(x+p_1) - w(x+p_2+2\Delta) + w(x+\Delta, \Delta)]$. By the contradicting assumption, $p_2 + \Delta \leq p_1 - \Delta$ and $p_2 + 2\Delta \leq p_1$. Hence, $b_1 > a_1$ and $b_2 > a_2$. Then strict concavity of u implies $\frac{u(a_2)-u(a_1)}{a_2-a_1} > \frac{u(b_2)-u(b_1)}{b_2-b_1}$. This inequality, $p_2 + 2\Delta \leq p_1$, and concavity of w imply that the left side of (26) exceeds the left side of (27). Then by (26) and (27), $w(m - p_2, \Delta) > w(m - p_1 + \Delta, \Delta)$. But this contradicts $p_2 < p_1 - \Delta$ and concavity of w.

(iii)Let $a = \max\{x_2 - x_1, m_1 - m_2\}$ and let $p_1 = \min p(x_1, m_1)$. Let $p_2 = a + p_1$. Assume that $x_2 > p_2$ and $m_2 + p_2 < B$; otherwise the result follows immediately. By $m_2 < m_1$ and $p_2 > p_1$, $w(m_2 + p_2, p_2) > w(m_1 + p_1, p_1)$. By $m_2 + p_2 \geq m_1 + p_1$, $w(m_2 + p_2 + \Delta, \Delta) \leq w(m_1 + p_1 + \Delta, \Delta)$. It follows that $\beta w(x_2 - p_2, \Delta) \geq \beta w(x_1 - p_1, \Delta) \geq u[\beta w(m_1 + p_1 + \Delta, p_1 + \Delta)] - u[\beta w(m_1 + p_1, p_1)] > u[\beta w(m_2 + p_2 + \Delta, p_2 + \Delta)] - u[\beta w(m_2 + p_2, p_2)]$, where the second inequality follows from $p_1 \in p(x_1, m_1)$ and the third from strict concavity of u. Note that $u[\beta w(m+p, p)] + \beta w(x-p)$, viewed as a function of p, is concave, and, hence, strictly increasing on $[0, \min p(x, m)]$ and strictly decreasing on $[\max p(x, m), \min\{x, B-m\}]$. Then $\beta w(x_2 - p_2, \Delta) > u[\beta w(m_2 + p_2 + \Delta, p_2 + \Delta)] - u[\beta w(m_2 + p_2, p_2)]$ implies $\max p(x_2, m_2) \leq p_2$.

(iv)First consider $x_2 > x_1$. Let $p = x_2 - x_1$. Assume that $x_1 > 0$ and $m_2 + p < B$; otherwise the result follows immediately. We have $\beta w(x_2 - p, \Delta) = \beta w(x_1, \Delta) \geq u[\beta w(m_1 + \Delta, \Delta)] - u(0) > u[\beta w(m_2 + p + \Delta, p + \Delta)] - u[\beta w(m_2 + p, p)]$, where the first inequality follows from $0 \in p(x_1, m_1)$ and the second from strict concavity of u. By the logic used in the proof of part (iii), $\max p(x_2, m_2) \leq p$. Next consider $x_2 = x_1$. Assume that $x_1 > 0$ and $m_2 < B$; otherwise the result follows immediately. We have $\beta w(x_2, \Delta) = \beta w(x_1, \Delta) \geq u[\beta w(m_1 + \Delta, \Delta)] \geq u[\beta w(m_2 + \Delta, \Delta)]$, where the first inequality follows from $0 \in p(x_1, m_1)$. By the logic used in the proof of part (iii), $\min p(x_2, m_2) = 0$.

(v)First, we claim that $u[\beta w(\Delta)] > \beta w(\Delta)$. Otherwise, by concavity of w, $u[\beta w(y+\Delta,\Delta)] \leq \beta w(\Delta)$ for $0 \leq y < B$, but because (w,π) is a steady state, this implies $w(\Delta) = 0$, a contradiction. By concavity of u and w, the claim implies for $x > m$, $u[\beta w(m+\Delta,\Delta)] > \beta w(m+\Delta,\Delta) \geq \beta w(x,\Delta)$. So $0 \notin p(x,m)$. ■

Proof of Lemma 10

Proof. We first claim that *if for each* $x > 0$*, there exists* m *such that* $p(x,m)$ *is a positive singleton, then* w *is strictly concave.*

The proof of the claim is by induction on the set satisfying strict concavity. That is, we show that w is strict concave on $\{0,\Delta,2\Delta\}$ and then show that strict concavity on $\{0,\Delta,...,x\}$ implies strict concavity on $\{0,\Delta,...,x,x+\Delta\}$. First, we prove that $2w(\Delta) > w(0)+w(2\Delta) = w(2\Delta)$. As shown in [10, Lemma 2, p. 967], $2f(x,m) \geq f(x-\Delta,m)+f(x+\Delta,m)$. Because $\pi(0) > 0$ and $f(0,0) = 0$, it suffices to show that $2f(\Delta,0) > f(2\Delta,0)$. By Lemma 6 (v), $f(\Delta,0) = u[\beta w(\Delta)] > \beta w(\Delta)$. There are two possibilities for $p(2\Delta,0)$. (i)If $\Delta \in p(2\Delta,0)$, then $f(2\Delta,0) = u[\beta w(\Delta)] + \beta w(\Delta) < 2u[\beta w(\Delta)] = 2f(\Delta,0)$. (ii)If $2\Delta \in p(2\Delta,0)$, then $f(2\Delta,0) = u[\beta w(2\Delta)] \leq u[2\beta w(\Delta)] < 2u[\beta w(\Delta)] = 2f(\Delta,0)$, where the first inequality follows from concavity of w and the second from strict concavity of u. Next for the induction step. Let m be such that $p(x,m)$ is a positive singleton. As above, because $\pi(m) > 0$, it suffices to show that $2f(x,m) > f(x-\Delta,m)+f(x+\Delta,m)$. Let $\min p(x-\Delta,m) = p$. By Lemma 6 (i), there are three possibilities for $\min p(x+\Delta,m)$. (i)$\min p(x+\Delta,m) = p$. By Lemma 6 (i), $\min p(x+\Delta,m) \geq \max p(x,m) \geq \min p(x-\Delta,m)$. It follows that $p(x,m) = \{p\}$ and $p \geq \Delta$. Therefore, $2f(x,m) - f(x-\Delta,m) - f(x+\Delta,m) = 2\beta w(x-p) - \beta w(x-\Delta-p) - \beta w(x+\Delta-p) > 0$, where the last inequality follows from $p \geq \Delta$ and the induction assumption. (ii)$\min p(x+\Delta,m) = p+\Delta$. Because $p, p+\Delta \in \Gamma(x,m)$ and $p(x,m)$ is a singleton, it follows that $2f(x,m) > u[\beta w(m+p+\Delta,p+\Delta)] + \beta w(x-p-\Delta) + u[\beta w(m+p,p)] + \beta w(x-p) = f(x-\Delta,m)+f(x+\Delta,m)$. (iii)$\min p(x+\Delta,m) = p+2\Delta$. Because $p+\Delta \in \Gamma(x,m)$, it follows that $2f(x,m) - f(x-\Delta,m) - f(x+\Delta,m) \geq 2u[\beta w(m+p+\Delta,p+\Delta)] - u[\beta w(m+p,p)] - u[\beta w(m+p+2\Delta,p+2\Delta)] > 0$, where the last inequality follows from strict concavity of u and concavity of w.

Now we can finish the proof of this lemma. By the claim, it suffices to prove that $\forall\, x > 0$, $\exists\, m^*$ such that $p(x,m^*)$ is a positive singleton. If $p(x,B-\Delta) = \{\Delta\}$, then $m^* = B-\Delta$ is as required. So assume $0 \in p(x,B-\Delta)$

and let $y = \max\{m : 0 \notin p(x, m)\}$. By Lemma 6 (ii), $p(x, y) = \{\Delta\}$. Then $m^* = y$. ■

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