Existence of a Monetary Steady State in a Matching Model: Divisible Money

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Abstract

Existence of a monetary steady state is established in a random matching model with divisible goods, divisible money, an arbitrary bound on individual money holdings, and take-it-or-leave-it offers by consumers. The monetary steady state shown to exist has nice properties: the value function, defined on money holdings, is strictly increasing and strictly concave, and the distribution over money holdings has full support. The approach is to show that the "limit" of the nice steady states for indivisible money, existence of which was established in an earlier paper, as the unit of money goes to zero is a monetary steady state for divisible money. For indivisible money, the marginal utility of consumption at zero was assumed to be large; for divisible money it is assumed to be large and finite.

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Running Title: **Existence: Divisible Money**

1 Introduction

In this paper, I study a random matching model with divisible goods, divisible money, an arbitrary bound on individual holdings, and take-it-or-leave-it offers by consumers. The model is essentially the one set out by Trejos and Wright [9] and Shi [7], which, in turn, is bases on Kiyotaki and Wright [4]. I show that there exists a monetary steady state with *nice* properties: the value function, defined on money holdings, is strictly increasing and strictly concave, and the distribution over money holdings has full support. This extends the existence result for indivisible money established in a previous paper (Zhu [11]).

As is well known, a matching model that allows for a large set of individual money holdings generates endogenous heterogeneity of money holdings. That makes it challenging to prove existence of a steady state with nice properties. Most researchers simplify or avoid the endogenous heterogeneity by making special assumptions.[1] In [11], I develop a technique that is built on the existence result of Taber and Wallace [8] on indivisible commodity money, and I show that there exists a nice steady state for fiat money that is a limit of steady states for commodity money as the direct utility payoff of money vanishes. That existence result is used in the current paper.[2]

The transition from indivisible money to divisible money is not trivial. My approach is as follows. I embed the nice steady states for indivisible money in the spaces of value functions and measures for divisible money. I then let the unit of indivisible money go to zero and show that a limit of the embedded steady states is a monetary steady state for divisible money. To carry through this approach, I need (a) existence of the limit and (b) continuity of a mapping whose fixed point is a divisible-money steady state. For existence of the limit, I equip the spaces of value functions and measures with some weak topologies. Although the topologies are weak, the mapping is continuous if the space of value functions is restricted to functions that

[1] For a review of literature, see footnote 1 in [11].

[2] From the result in [11], one can easily construct a class of steady states for divisible money with finite support $\{0, p, 2p, ...B\} \equiv X$, where B is the bound on individual money holdings and B/p is an integer. For such a steady state, the value function is a step function with jumps at the support and the restriction of the value function and the distribution to X is a nice indivisible-money steady state with p as the smallest unit of indivisible money. Because the focus of this paper is steady states with nice properties, such a construction is not used.

are continuous, concave, and strictly increasing. Therefore, the main task is to show that the limit value function is in that space. The challenging property is continuity at zero. To establish continuity, I assume that the marginal utility of consumption at zero is finite, and, show, through a lengthy argument, that the slope of indivisible-money steady-state value functions is uniformly bounded.

The proof technique used here is applicable to other versions of matching models of money—for example, the alternating endowment model studied in Wallace and Zhu [10], and the random matching model with money creation studied in Molico [6]. It could also be useful in other models with endogenous heterogeneity.

2 The model and the existence result

The model is that in [11], except that money is now divisible.

2.1 Environment

Time is discrete, dated as $t \geq 0$. There is a $[0, 1]$ continuum of each of $N \geq 3$ types of infinitely lived agents, and there are N distinct produced and perishable types of divisible goods at each date. A type n agent, $n \in \{1, 2, ..., N\}$, produces only good n and consumes only good $n + 1$ (modulo N). Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. For a type n agent, utility in a period is $u(q_{n+1}) - q_n$, where $q_{n+1} \in \mathbb{R}_+$ is the amount of good $n+1$ consumed and $q_n \in \mathbb{R}_+$ is the amount of good n produced. The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, and continuously differentiable and satisfies $u(0) = 0$ and $u'(\infty) = 0$. In addition, I assume that $u'(0)$ is large but finite. The lower bound on $u'(0)$ is specified below. Finiteness of $u'(0)$ is used only to establish a uniform upper bound on the slope of the indivisible-money value functions.

There exists a fixed stock of money which is perfectly durable and divisible. Money is symmetrically distributed across the N specialization types. There is an exogenous (finite) upper bound on individual holdings. The bound is denoted by B. I assume that B is large relative to the average holding. Note that B larger than the average holding is necessary for trade to occur. Further, I normalize the average money holding per specialization type to be unity.

In each period, agents are randomly matched in pairs. A meeting between a type n agent, a consumer, and a type $n+1$ agent, a producer, is called a single-coincidence meeting. Other meetings are not relevant. In meetings, agents' types and money holdings are observable, but any other information about an agent's trading history is private. In a meeting, the consumer makes a take-it-or-leave-it offer to the producer.

2.2 Definition of equilibrium

The definition of equilibrium for divisible money is analogous to that for indivisible money in [11]. As in [11], equilibria in consideration are symmetric over agent types.

Let $v : [0, B] \to \mathbb{R}_+$ be non decreasing. If v is taken to be the value function defined on money holdings at the start of the next period, prior to the realization of matching, then payoffs and optimal offers in meetings of the current period can be defined in terms of v. In a meeting between a consumer with holding x (pre-trade) and a producer with holding m (pre-trade), the consumer may choose to pay p amount of money to the producer, where $p \in \Gamma(x, m) \equiv [0, \min\{x, B - m\}]$, the set of feasible payments of money. Take-it-or-leave-it offers by the consumer imply that the consumer can demand production equal to $\beta[v(m + p) - v(m)]$ if he pays p. It is convenient to define $v(a_2, a_1) \equiv v(a_2) - v(a_2 - a_1)$. Let

$$\tilde{f}(x, m, v) = \max_{p \in \Gamma(x,m)} u[\beta v(m + p, p)] + \beta v(x - p). \tag{1}$$

Also, let

$$\tilde{p}(x, m, v) \in \arg \max_{p \in \Gamma(x,m)} u[\beta v(m + p, p)] + \beta v(x - p) \tag{2}$$

be such that $\tilde{p}(., ., v) : [0, B]^2 \to [0, B]$ is Borel measurable. Hence, $\tilde{f}(x, m, v)$ is the payoff for the consumer and $\tilde{p}(x, m, v)$ is the optimal offer of money. (It is important to remember that the first argument of $\tilde{f}$ and $\tilde{p}$ is the consumer's pre-trade holding and the second is the producer's.) To express the law of motion for the distribution of money holdings, I need some additional notation. Let the set-valued mappings $\gamma(.; v)$ and $\zeta(.; v)$ on the Borel sets of $[0, B]$ be defined by

$$\begin{aligned} \gamma([0, x]; v) &= \{(y, m) \in [0, B]^2, y - \tilde{p}(y, m, v) \leq x\}, & (3) \\ \zeta([0, x]; v) &= \{(m, y) \in [0, B]^2, y + \tilde{p}(m, y, v) \leq x\}. & (4) \end{aligned}$$

Here, $\gamma([0,x];v)$ is the set of pairs of (y,m) such that if a consumer with y (pre-trade) meets a producer with m (pre-trade), then the consumer's post-trade holding is no greater than x; and $\zeta([0,x];v)$ is the set of pairs of (m,y) such that if a consumer with m (pre-trade) meets a producer with y (pre-trade), then the producer's post-trade holding is no greater than x. Moreover, for a Borel measure μ on $[0,B]$, let μ^2 denote the product Borel measure $\mu \times \mu$ on $[0,B]^2$.

Now let v_t denote the value function of money holdings at the start of period t, and let μ_t denote the Borel measure of money holdings on $[0,B]$ at the start of period t, so that $\mu_t[0,x]$ is the fraction of agents holding money no greater than x before period t matching. Then the value function v_t satisfies

$$v_t(x) = \frac{N-1}{N}\beta v_{t+1}(x) + \frac{1}{N}\int \tilde{f}(x,m,v_{t+1})d\mu_t(m). \tag{5}$$

This follows from the fact that the payoff to being a producer with x is $\beta v_{t+1}(x)$. And the law of motion for μ_{t+1} can be expressed as

$$\mu_{t+1}[0,x] = \frac{N-2}{N}\mu_t[0,x] + \frac{1}{N}\mu_t^2\gamma([0,x];v_{t+1}) + \frac{1}{N}\mu_t^2\zeta([0,x];v_{t+1}). \tag{6}$$

Here, $\mu_t^2\gamma([0,x];v_{t+1})$ is the measure of consumers with post-trade holdings no greater than x and $\mu_t^2\zeta([0,x];v_{t+1})$ is the measure of producers with post-trade holdings no greater than x.

Definition 1 *Given μ_0, a sequence $\{v_t,\mu_{t+1}\}_{t=0}^{\infty}$ is an equilibrium if it satisfies $(1)-(6)$. A monetary equilibrium is an equilibrium with positive consumption and production. A pair (v,μ) is a steady state if $\{v_t,\mu_{t+1}\}_{t=0}^{\infty}$ with $v_t = v$ and $\mu_{t+1} = \mu$ is an equilibrium for $\mu_0 = \mu$.*

2.3 The main result

The main result is the following theorem.

Theorem 1 *If $B \geq 8$ and $\infty > u'(0) \geq [4/(R\beta)]^2$, where $R \equiv [(N-(N-1)\beta]^{-1}$, then there exists a monetary steady state, (v,μ), where v is continuous, strictly increasing, and strictly concave and μ has full support.*

To establish the theorem, one conceivable approach is to directly mimic the proof for indivisible money in [11, Proposition 1]: (i) attach some utility to divisible money and prove that a steady state exists; (ii) take a limit as the utility goes to zero and show that the limit is bounded away from the non monetary steady state. Step (i) would seemingly appeal to Fan's Fixed Point Theorem (see [1, p. 550]). This involves defining a mapping whose fixed point is a steady state. The hypotheses include: (a) the mapping is continuous; (b) the domain of the mapping is compact; (c) the image of the mapping is contained in the domain. However, they cannot be satisfied for divisible money. Therefore, I pursue another approach. I start with the indivisible-money steady state (for fiat money) and let the size of the smallest unit approach zero. The next three sections are devoted to carrying out that approach. Although the general strategy is straightforward, the details are not simple.

In section 3, I formally define the mapping implied by (1)-(6) and construct a candidate for the steady state. The domain of the mapping is $\mathbf{V} \times \mathbf{\Lambda}$, where $\mathbf{V}$ is a set of bounded, continuous, concave, and strictly increasing real functions defined on $[0, B]$, and $\mathbf{\Lambda}$ is the set of probability measures defined on $[0, B]$ with unit mean. Here, $\mathbf{V}$ is taken as a subspace of the space of all real functions defined on $[0, B]$ with the product topology, and $\mathbf{\Lambda}$ is taken as a subspace of all probability measures on $[0, B]$ with the weak* topology. To construct the candidate, I embed the nice steady states for indivisible money in $\mathbf{V} \times \mathbf{\Lambda}$. In particular, a steady-state value function is embedded using linear interpolation. Then, I let the unit of indivisible money go to zero and take a limit of the embedded steady states. The limit, denoted (v, μ), is the candidate, where v is in the closure of $\mathbf{V}$. In fact, I show that v is strictly increasing. Next, I show that if $v \in \mathbf{V}$, then (v, μ) is a fixed point of the mapping. Here, I use continuity of the mapping. I also use the following result: if the unit of money is close to zero, then the embedded steady state is close to its image under the mapping. This is the exact intuition behind the approximation argument. The main remaining challenge is showing that $v \in \mathbf{V}$ (that v is continuous at 0). As is well-known, $\mathbf{V}$ is not complete in that sense.

Section 4 is devoted to establishing an intermediate result. In order to show that v is continuous at 0, I use finiteness of $u'(0)$ to establish that the slope of the embedded value functions is uniformly bounded. The proof of boundedness proceeds by way of contradiction. By the contradicting assumption, if the unit of money, Δ, is sufficiently small, then in a nice indivisible-money steady state, the marginal value of money at Δ is arbitrarily large. This and $u'(0) < \infty$ can be shown to imply that almost all agents must have money holdings near zero, which is impossible.

Roughly speaking, the argument in section 4 leading to the contradiction proceeds as follows. For a nice indivisible-money steady state, assume by contradiction that the marginal value at money holding x_1 is large. Because the value function is bounded independently of Δ, the product of x_1 and the marginal value at x_1 is bounded. Therefore, x_1 is small. Because $u'(0)$ is finite, consumers with holdings no greater than x_1 only trade with producers with holdings no greater than some x_2, where the marginal value at x_2 is also large. It follows that x_2 is small. However, producers with holdings no greater than x_1 trade with all consumers with holdings greater than x_1. Because the outflow from and inflow into $\{0, ..., x_1\}$ are equal, it follows that the measure of the set $\{x_1, ..., x_2\}$, or a close approximation to it, is bounded from below. By an elaborate induction argument, this leads to the contradiction. While section 4 is the innovative part of the proof of the main theorem, the flow of the argument in the rest of the paper can be followed provided the uniform boundedness conclusion, the conclusion of Proposition 1, is accepted.

In section 5, I first use the uniform bound on the slope of the embedded value functions to prove Theorem 1 except for the full support property. Then I complete the proof by showing that the steady state measure has full support.

3 A candidate for a monetary steady state

I start by defining the mapping implied by (1)-(6). Let W be the unique positive solution of $N(1-\beta)W = u(\beta W)+N$. Let $\mathbf{V}$ be the set of continuous, concave, and strictly increasing functions from $[0, B]$ to $[0, W]$ and let $\mathbf{\Lambda}$ be the set of Borel probability measures on $[0, B]$ satisfying the unit mean condition. Here, $\mathbf{V}$ is taken as a subspace of the space of all real functions defined on $[0, B]$ with the product topology, and $\mathbf{\Lambda}$ is taken as a subspace of all probability measures on $[0, B]$ with the weak* topology (see [1, p. 50 and p. 474]). Let the mapping $T = (T_v, T_\mu)$ on $\mathbf{V} \times \mathbf{\Lambda}$ be defined by

$$T_v(v, \mu)(x) = \frac{N-1}{N}\beta v(x) + \frac{1}{N}\int \tilde{f}(x, m, v)d\mu(m), \tag{7}$$

and

$$T_\mu(v, \mu)[0, x] = \frac{N-2}{N}\mu[0, x] + \frac{1}{N}\mu^2\gamma([0, x]; v) + \frac{1}{N}\mu^2\zeta([0, x]; v). \tag{8}$$

By Definition 1, a fixed point of T is a monetary steady state.

As introduced above, a candidate for a fixed point of T is to be constructed. The construction uses results set up in the following lemmas. Lemma 1 establishes continuity of T. Lemma 2 gives existence of nice indivisible-money steady states. Then I embed Lemma 2 indivisible-money steady states in $\mathbf{V} \times \mathbf{\Lambda}$. Lemma 3 shows that a subsequence of the embedded steady states converges to some limit point as the unit of money goes to zero. The limit point is the candidate. In fact, Lemma 3 shows that the limit value function is strictly increasing. Finally, Lemma 4 shows that if the limit value function is in $\mathbf{V}$, then the limit point is indeed a monetary steady state.

The next lemma establishes continuity of T.

Lemma 1 $T : \mathbf{V} \times \mathbf{\Lambda} \to \mathbf{V} \times \mathbf{\Lambda}$ *is continuous.*

Proof. First, we note that $\mathbf{V} \times \mathbf{\Lambda}$ is a metric space. It is well known that pointwise convergence in $\mathbf{V}$ implies uniform convergence (for instance, see [5, Exercise 2, p. 86]). Hence $\mathbf{V}$ is metrizable. By [1, 14.11 Theorem, p. 482], $\mathbf{\Lambda}$ is metrizable. So $\mathbf{V} \times \mathbf{\Lambda}$ is metrizable.

Next, we show that $\tilde{f}(.,.,.)$ and $\tilde{p}(.,.,.)$ are continuous on $[0, B]^2 \times \mathbf{V}$. To see this, let $A = \{(x, m, v, p) : (x, m, v) \in [0, B]^2 \times \mathbf{V}$ and $p \in \Gamma(x, m)\}$, and let $k : A \to \mathbb{R}_+$ be defined by $k(x, m, v, p) = u[\beta v(m + p, p)] + \beta v(x - p)$. Because pointwise convergence in $\mathbf{V}$ implies uniform convergence, k is continuous on A. Then because the correspondence $(x, m, v) \mapsto \Gamma(x, m)$ is continuous, it follows from Berge's Maximum Theorem (see [1, p. 539]) that $\tilde{f}$ is continuous and $\tilde{p}$ is upper hemicontinuous. For $(x, m, v) \in [0, B]^2 \times \mathbf{V}$, because v is concave and strictly increasing, $u[\beta v(m + p, p)] + \beta v(x - p)$, viewed as a function of p, is strictly concave. Hence $\tilde{p}(x, m, v)$ is a singleton and so $\tilde{p}$ is continuous.

Next, we show that T is a single-valued mapping from $\mathbf{V} \times \mathbf{\Lambda}$ to itself. Fix $(v, \mu) \in \mathbf{V} \times \mathbf{\Lambda}$. By continuity of $\tilde{f}$ and $\tilde{p}$, $\int \tilde{f}(x, m, v) d\mu(m)$, $\gamma([0, x]; v)$, and $\zeta([0, x]; v)$ are well defined for all x. Hence $T(v, \mu)$ is well defined. As shown above, $\tilde{p}(x, m, v)$ is a singleton for $(x, m) \in [0, B]^2$. Hence $T(v, \mu)$ is single-valued. Clearly, $T_\mu(v, \mu) \in \mathbf{\Lambda}$, and $T_v(v, \mu)$ is strictly increasing with $T_v(v, \mu)(x) \in [0, W]$ for $x \in [0, B]$. By continuity of $\tilde{f}$ and the Dominated Convergence Theorem (see [1, p. 407]), $\int \tilde{f}(x_n, m, v) d\mu(m) \to \int \tilde{f}(x, m, v) d\mu(m)$ as $x_n \to x$. Hence $T_v(v, \mu)$ is continuous. Now let $0 \leq x_1 < x_2 \leq B$, $0 < \alpha < 1$, and $x = \alpha x_1 + (1 - \alpha) x_2$. Because $\alpha \tilde{p}(x_1, m, v) + (1 - \alpha) \tilde{p}(x_2, m, v) \in [0, x]$, it follows that $\tilde{f}(x, m, v) \geq \alpha \tilde{f}(x_1, m, v) + (1 - \alpha) \tilde{f}(x_2, m, v)$. Hence $T_v(v, \mu)$ is concave. This gives $T(v, \mu) \in \mathbf{V} \times \mathbf{\Lambda}$.

Next, we claim that *if* $v_n \to v$, *then* $\tilde{f}(.,.,v_n)$ *and* $\tilde{p}(.,.,v_n)$ *converge uniformly to* $\tilde{f}(.,.,v)$ *and* $\tilde{p}(.,.,v)$, *respectively.* It suffices to show that $\tilde{f}(.,.,.)$ and $\tilde{p}(.,.,)$ are uniformly continuous on $[0,B]^2 \times \{v, v_1, v_2, ...\}$. This follows because $\tilde{f}$ and $\tilde{p}$ are continuous, $\{v, v_1, v_2, ...\}$ is compact (see [1, 2.35 Theorem, p. 41]), and continuity in a compact domain implies uniform continuity (see [5, (4.1) Theorem , p. 79]).

Next, for $v \in \mathbf{V}$, let $g_v, h_v : [0,B]^2 \to [0,B]$ be defined by $g_v(x,m) = x - \tilde{p}(x,m,v)$ and $h_v(x,m) = x + \tilde{p}(m,x,v)$. Note that $\gamma([0,x];v) = g_v^{-1}[0,x]$ and $\zeta([0,x];v) = h_v^{-1}[0,x]$.

Now let $(v_n, \mu_n) \to (v, \mu)$. For continuity of T_μ, it suffices to show that $\mu_n^2 g_{v_n}^{-1}$ and $\mu_n^2 h_{v_n}^{-1}$ converge weakly to $\mu^2 g_v^{-1}$ and $\mu^2 h_v^{-1}$, respectively. This follows from [2, Theorem 5.5, p. 34], the claim, and that μ_n^2 converges weakly to μ^2. For continuity of T_v, it suffices to show that $\int \tilde{f}(x,m,v_n)d\mu_n(m) \to \int \tilde{f}(x,m,v)d\mu(m)$ for $x \in [0,B]$. This follows from [1, 14.7 Corollary, p. 480] and the claim. ■

As noted above, my approach is to approximate a steady state for divisible money using steady states for indivisible money. To begin with, I introduce the relevant notation and definitions. Let the unit of indivisible money be denoted by Δ, the set $\{0, \Delta, 2\Delta, ..., B\}$ by B_Δ, and the set $\{p \in B_\Delta, p \leq \min\{x, B-m\}\}$ by $\Gamma_\Delta(x,m)$. That is, B_Δ is the indivisible-money counterpart of $[0,B]$ and $\Gamma_\Delta(x,m)$ is that of $\Gamma(x,m)$. Let (w_Δ, π_Δ) be a steady state for indivisible money with Δ as the unit of money, where w_Δ is the value function of money holdings and π_Δ is the measure of money holdings. That is, (w_Δ, π_Δ) is the indivisible-money counterpart of (v, μ). Also, let $w_\Delta(a_2, a_1) \equiv w_\Delta(a_2) - w_\Delta(a_2 - a_1)$. Let

$$f(x,m,w_\Delta) = \max_{p \in \Gamma_\Delta(x,m)} u[\beta w_\Delta(m+p,p)] + \beta w_\Delta(x-p), \tag{9}$$

$$p(x,m,w_\Delta) = \arg\max_{p \in \Gamma_\Delta(x,m)} u[\beta w_\Delta(m+p,p)] + \beta w_\Delta(x-p). \tag{10}$$

Note that f is the indivisible-money counterpart of $\tilde{f}$ in (1) and p is that of $\tilde{p}$ in (2). For a measure π on B_Δ and an interval I, let $\pi I \equiv \pi(I \cap B_\Delta)$. Finally, a real function w defined on B_Δ is concave if $2w(x) \geq w(x-\Delta) + w(x+\Delta)$ for $0 < x < B$.

The next lemma is the indivisible-money counterpart of Theorem 1.

Lemma 2 *If* $B \geq 8$ *and* $u'(0) \geq [4/(R\beta)]^2$, *then there exists a monetary steady state* (w_Δ, π_Δ) *with* w_Δ *strictly increasing and strictly concave and* *with* $w_\Delta(0) = 0$ *and* $D/\beta \leq w_\Delta(4) < W$*, where* D *is the unique solution of* $u'(D) = [4/(R\beta)]^2$*.*

Proof. See [11, Proposition 1]. ■

For a Lemma 2 steady state (w_Δ, π_Δ), let v_Δ be the linear interpolation of w_Δ and let μ_Δ be the extension of π_Δ to $[0, B]$. That is, if $b \in B_\Delta$ and $x \in [b, b + \Delta)$, then

$$v_\Delta(x) \equiv w_\Delta(b) + (x - b)w_\Delta(b + \Delta, \Delta)/\Delta, \tag{11}$$

$$\mu_\Delta\{b\} = \mu_\Delta[b, x) \equiv \pi_\Delta(b). \tag{12}$$

By definition, if w defined on B_Δ is concave, then its linear interpolation on $[0, B]$ is concave. Hence, (v_Δ, μ_Δ) is an embedding of (w_Δ, π_Δ) in $\mathbf{V} \times \mathbf{\Lambda}$. It is convenient to choose $\Delta = 10^{-n}B$ for $n \in \mathbb{N}$ as the sequence of the units of indivisible money, and let the corresponding sequence of embedded steady states be denoted by $\{(v_\Delta, \mu_\Delta)\}_\Delta$.

The next lemma gives a candidate for a monetary steady state.

Lemma 3 *The sequence* $\{(v_\Delta, \mu_\Delta)\}_\Delta$ *has a limit point* (v, μ)*, where* v *is in the closure of* $\mathbf{V}$ *and is concave and strictly increasing, and* μ *is in* $\mathbf{\Lambda}$*.*

Proof. By [1, 14.11 Theorem, p. 482], the closure of $\mathbf{\Lambda}$ is compact. Because $\mathbf{\Lambda}$ is closed, it follows that $\mathbf{\Lambda}$ is compact. By the Tychonoff Product Theorem (see [1, p. 52]), $\bar{\mathbf{V}}$ (the closure of $\mathbf{V}$) is compact. Hence $\bar{\mathbf{V}} \times \mathbf{\Lambda}$ is compact, and therefore, $\{(v_\Delta, \mu_\Delta)\}_\Delta$ has a limit point (v, μ) in $\bar{\mathbf{V}} \times \mathbf{\Lambda}$. Concavity of v is obvious. For strict monotonicity, see the appendix. ■

The next lemma shows that a Lemma 3 limit point (v, μ) is a monetary steady state if $v \in \mathbf{V}$.

Lemma 4 *Let* (v, μ) *be a Lemma* 3 *limit point. If* $v \in \mathbf{V}$*, then* $T(v, \mu) = (v, \mu)$*.*

Proof. Let $\{(v_\Delta, \mu_\Delta)\}_\Delta$ be a sequence that converges to (v, μ) and let $\{(w_\Delta, \pi_\Delta)\}_\Delta$ be the sequence of Lemma 2 steady states corresponding to $\{(v_\Delta, \mu_\Delta)\}_\Delta$. Let $T_v(v_\Delta, \mu_\Delta)$ be denoted by Tv_Δ and $T_\mu(v_\Delta, \mu_\Delta)$ by $T\mu_\Delta$. Let $d(.,.)$ be a metric on $\mathbf{V} \times \mathbf{\Lambda}$. Lemma 1(ii) implies $d[(Tv_\Delta, T\mu_\Delta), T(v, \mu)] \rightarrow$

0. Because $d[(v,\mu),T(v,\mu)] \le d[(v,\mu),(Tv_\Delta,T\mu_\Delta)]+d[(Tv_\Delta,T\mu_\Delta),T(v,\mu)]$, it suffices to prove (i)$Tv_\Delta \to v$ and (ii)$T\mu_\Delta \to \mu$.

(i) Note that $Tv_\Delta(0)=v(0)=0$. So it suffices to show $Tv_\Delta(x)\to v(x)$ for $x>0$. Fix $x>0$ and fix $\varepsilon>0$. Let Δ be small such that $|v_\Delta(y)-v(y)|<\varepsilon$ all y and $v(\Delta)<\varepsilon$. The first inequality holds for small Δ because pointwise convergence in $\mathbf{V}$ implies uniform convergence. The second inequality and concavity imply $|v(y)-v(z)|<\varepsilon$ for $|y-z|\le\Delta$. First consider $x\in B_\Delta$. By construction (see (11)),

$$v_\Delta(x)=\frac{N-1}{N}\beta v_\Delta(x)+\frac{1}{N}\textstyle\sum_{m\in B_\Delta}\mu_\Delta\{m\}\bar{f}(x,m,v_\Delta),$$

where

$$\bar{f}(x,m,v_\Delta)=\max_{p\in\Gamma_\Delta(x,m)} u[\beta v_\Delta(m+p,p)]+\beta v_\Delta(x-p). \tag{13}$$

By definition (see (7)),

$$Tv_\Delta(x)=\frac{N-1}{N}\beta v_\Delta(x)+\frac{1}{N}\textstyle\sum_{m\in B_\Delta}\mu_\Delta\{m\}\tilde{f}(x,m,v_\Delta).$$

Fix (x,m) and let $c=\tilde{f}(x,m,v_\Delta)-\bar{f}(x,m,v_\Delta)$. Note that $c\ge 0$. Let $p=\tilde{p}(x,m,v_\Delta)$. A lower bound on $\bar{f}(x,m,v_\Delta)$ can be obtained by taking $p'\in[p,p+\Delta]\cap B_\Delta$ in (13). Now $p'\ge p$ implies $c\le\beta[v_\Delta(x-p)-v_\Delta(x-p')]$, and this and $|v_\Delta(y)-v(y)|<\varepsilon$ all y imply $c<v(x-p)-v(x-p')+2\varepsilon$. Then it follows from $p'-p\le\Delta$ that $c<3\varepsilon$. Hence $0\le Tv_\Delta(x)-v_\Delta(x)<3\varepsilon/N$. By $|Tv_\Delta(x)-v(x)|\le|Tv_\Delta(x)-v_\Delta(x)|+|v_\Delta(x)-v(x)|$,

$$|Tv_\Delta(x)-v(x)|<3\varepsilon/N+\varepsilon \text{ for } x\in B_\Delta. \tag{14}$$

Next consider $x\notin B_\Delta$. Let $a_0=\max\{m\in B_\Delta : m<x\}$ and $a_1=a_0+\Delta$. Monotonicity of Tv_Δ implies $|Tv_\Delta(x)-v(x)|<\max_{i=0,1}|Tv_\Delta(a_i)-v(x)|$, and it follows from (14) and $|a_i-x|<\Delta$ that for $i=0,1$,

$$|Tv_\Delta(a_i)-v(x)|\le|Tv_\Delta(a_i)-v(a_i)|+|v(a_i)-v(x)|<3\varepsilon/N+2\varepsilon.$$

This gives $Tv_\Delta(x)\to v(x)$.

(ii) First we introduce some notation. For each $(y,m)\in B_\Delta^2$, let

$$\bar{p}(y,m,v_\Delta)=\arg\max_{p\in\Gamma_\Delta(y,m)} u[\beta v_\Delta(m+p,p)]+\beta v_\Delta(y-p).$$

(Note that $\bar{p}(y,m,v_\Delta)$ may be multi-valued.) By construction (see (12)),

$$\mu_\Delta[0,x] = \frac{N-2}{N}\mu_\Delta[0,x] + \frac{1}{N}\bar{\gamma}_\Delta[0,x] + \frac{1}{N}\bar{\zeta}_\Delta[0,x],$$
$$\bar{\gamma}_\Delta[0,x] = \textstyle\sum_{(y,m)\in B_\Delta^2} \bar{\lambda}_\Delta([0,x];y,m)\mu_\Delta\{y\}\mu_\Delta\{m\},$$
$$\bar{\zeta}_\Delta[0,x] = \textstyle\sum_{(m,y)\in B_\Delta^2} \bar{\lambda}_\Delta([0,x];m,y)\mu_\Delta\{m\}\mu_\Delta\{y\},$$

where $\bar{\lambda}_\Delta(.;y,m)$ is a probability measure on $[0,B]$ satisfying

$$\bar{\lambda}_\Delta(A;y,m) = 1 \text{ where } A = \{z : z \in y - \bar{p}(y,m,v_\Delta)\}.$$

By definition (see (8)),

$$T\mu_\Delta[0,x] = \frac{N-2}{N}\mu_\Delta[0,x] + \frac{1}{N}\tilde{\gamma}_\Delta[0,x] + \frac{1}{N}\tilde{\zeta}_\Delta[0,x],$$
$$\tilde{\gamma}_\Delta[0,x] = \textstyle\sum_{(y,m)\in B_\Delta^2} \tilde{\lambda}_\Delta([0,x];y,m)\mu_\Delta\{y\}\mu_\Delta\{m\},$$
$$\tilde{\zeta}_\Delta[0,x] = \textstyle\sum_{(m,y)\in B_\Delta^2} \tilde{\lambda}_\Delta([0,x];m,y)\mu_\Delta\{m\}\mu_\Delta\{y\},$$

where $\tilde{\lambda}_\Delta(.;y,m)$ is the probability measure on $[0,B]$ satisfying

$$\tilde{\lambda}_\Delta(\{z\};y,m) = 1 \text{ where } z = y - \tilde{p}(y,m,v_\Delta).$$

Next, we introduce a metric to metricize the weak* topology of $\mathbf{\Lambda}$. The metric d_L, called Lévy distance (see Huber [3, p. 25]), is defined by

$$\begin{aligned} d_L(\mu_1,\mu_2) &= \inf\{\varepsilon : \forall x, \\ \mu_1[0,\min\{0,x-\varepsilon\}] - \varepsilon &\le \mu_2[0,x] \le \mu_1[0,\max\{B,x+\varepsilon\}] + \varepsilon\}. \end{aligned}$$

Note that for $(y,m) \in B_\Delta^2$ and $p \in \bar{p}(y,m,v_\Delta)$, $|p - \tilde{p}(y,m,v_\Delta)| \le \Delta$. It follows that

$$\forall x,\ \bar{\gamma}_\Delta[0,\min\{0,x-\Delta\}] \le \tilde{\gamma}_\Delta[0,x] \le \bar{\gamma}_\Delta[0,\max\{B,x+\Delta\}].$$

Hence $d_L(\tilde{\gamma}_\Delta,\bar{\gamma}_\Delta) \le \Delta$. Similarly, $d_L(\tilde{\zeta}_\Delta,\bar{\zeta}_\Delta) \le \Delta$. This gives $T\mu_\Delta \to \mu$. ■

4 A uniform upper bound on the slope of indivisible-money value functions

Let (v,μ) be a Lemma 3 limit point. By Lemma 4, to prove existence of a monetary steady state, it suffices to show that v is continuous. To this end, I use the assumption that $u'(0)$ is finite. In the rest of this section, let $\{(v_\Delta, \mu_\Delta)\}_\Delta$ be a sequence that converges to (v, μ), and let $\{(w_\Delta, \pi_\Delta)\}_\Delta$ be the corresponding sequence of Lemma 2 steady states. Using the finiteness assumption of $u'(0)$, I show a stronger result: $w_\Delta(\Delta)/\Delta$ is uniformly bounded.

The argument requires considerable additional notation. In order to facilitate subsequent reference to it, I group the notation in a definition.

Definition 2 *(i) For $x_0 \in B_\Delta \backslash \{0\}$, let the sequence $\{x_n, y_n, z_n\}_{n\geq 1}$ be defined by*

$$
\begin{aligned}
x_n &= \max\{x : p(x_{n-1}, x, w_\Delta) \neq \{0\}\} + \Delta, \\
z_n &= \min\{x : \min p(x, 0, w_\Delta) \geq x_n\}, \\
y_n &= \max\{x_n + x_{n+1}, z_n\}.
\end{aligned}
$$

(ii) Let $\rho \equiv \frac{N(1-\beta)}{[u'(0)-1]\beta}$. Fix a continuity point $\hat{x}$ of μ with $\hat{x} \geq 1$, and let $L \equiv \min\{n \in \mathbb{N} : 2^{-n+1} \leq \mu[\hat{x}, B])\}$ and $K \equiv \sum_{j=1}^{L} 2^{j-1}$. For the quadratic equation $x^2 + s_1 x - s_2 = 0$ with $(s_1, s_2) > (0, 0)$, denote its unique positive root by $g(s_1, s_2)$. Let $\hat{g} \equiv \min\{g(s_1, s_2) : (s_1, s_2) \in [\rho, 1] \times [\rho^2/K, 1]\}$, and let $\sigma(l) = int(2^{-l}/\hat{g})$, where $int(x)$ is the smallest integer no less than x.

(iii) Let $\omega_0 = \Delta$, and let the sequence $\{\omega_l\}_{l\geq 1}$ be defined by $\omega_l = y_{\sigma(l)}$, where $y_{\sigma(l)}$ is determined by the part (i) sequence $\{(x_n, y_n, z_n)\}$ for $x_0 = \omega_{l-1}$.

In part (i) of the definition, x_n is the money holding of the poorest producers with whom consumers with x_{n-1} do not trade, while z_n is that of the poorest consumers who offer at least x_n to producers with 0. In part (ii), the restriction $\hat{x} \geq 1$ is without loss of generality. Also note that $\hat{g} > 0$.

As noted above, the proof of the uniform boundedness is by way of contradiction, and the contradiction is achieved by an induction argument to show that almost all agents must have money holdings near zero. To elaborate further, the induction argument follows two related lines. The first line is to show that the sequence $\{\omega_l\}$ in Definition 2 is a measure-exhausting sequence of money holdings, where measure exhausting means that for some term in the sequence, the measure of all larger holdings is bounded above in a useful way. Because ω_l is determined by ω_{l-1} through the sequence $\{x_n, y_n, z_n\}$ for $x_0 = \omega_{l-1}$ in Definition 2, not surprisingly, the measure-exhausting property of $\{\omega_l\}$ is derived from the same property of $\{x_n, y_n, z_n\}$. The main lemmas here are Lemmas 7 and 8. The main analysis here uses the outflow-equal-inflow properties of a steady state. The second line of the argument pertains to existence of ω_l for small Δ and for large l. (It turns out that $l = L$ is sufficiently large.) The main lemmas here are Lemmas 10 and 11. The key here is to find a link between the marginal values of money at ω_l and ω_{l-1}. The link and the unbounded marginal value at Δ guarantee the existence. A crucial intermediate step, Lemma 17, is stated and proved in the appendix. In Proposition 1, the two lines are combined to draw the contradiction.

In the next two lemmas, I collect some preliminary results. Lemma 5 describes the dependence of the optimal offer in (10) on the money holdings of the consumer and the producer. Lemma 6 gives some basic properties of $\{(x_n, y_n, z_n)\}$.

Lemma 5 *Let* $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$.

(i) If $x > y$, *then* $0 \notin p(x, y, w_\Delta)$.

(ii) If $p_1 \in p(x, m, w_\Delta)$ *and* $p_2 \in p(x + \Delta, m, w_\Delta)$, *then* $p_2 - p_1 \in \{0, \Delta\}$.

(iii) If $p_1 \in p(x, m, w_\Delta)$ *and* $p_2 \in p(x, m + \Delta, w_\Delta)$, *then* $p_1 \leq p_2 + \Delta$.

(iv) If $0 \in p(x', m', w_\Delta)$, $x > x'$, *and* $m \geq m' - (x - x')$, *then* $\max p(x, m, w_\Delta) \leq x - x'$.

Proof. See [11, Lemma 6]. ■

Lemma 6 *Let* $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$. *Let* $\{(x_n, y_n, z_n)\}$ *for* $x_0 \in B_\Delta \backslash \{0\}$ *and* ρ *be as given in Definition 2.*

(i) $\{x_n\}$ *exists and is non decreasing.*

(ii) If $\{z_n\}$ *exists, then it is non decreasing.*

(iii) If $\{y_n\}$ *exists, then it is non decreasing.*

(iv) $\pi_\Delta[0, x_n) > \rho$ *for* $n \geq 1$.

Proof. See the appendix. ■

The following lemma establishes the key property of the sequence $\{(x_n, y_n, z_n)\}$. That is, it is a measure-exhausting sequence.

Lemma 7 *Let* $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$. *Let* $\{(x_n, y_n, z_n)\}$ *for* $x_0 \in B_\Delta \backslash \{0\}$, ρ, K, *and* $\hat{g}$ *be as given in Definition 2. Assume that* y_n *exists.*

(i) If $\pi_\Delta[y_n, B] - \pi_\Delta[x_{n+1}, y_n) < \frac{\rho}{K}$, *then* $\pi_\Delta[y_n, B] < \frac{1}{2}\{1 - \pi[0, x_n) + \frac{\rho}{K}\}$.

(ii) If $\pi_\Delta[y_n, B] - \pi_\Delta[x_{n+1}, y_n) \geq \frac{\rho}{K}$, *then* $x_{n+1} > x_n$ *and* $\pi_\Delta[x_n, x_{n+1}) \geq \hat{g}$.

Proof. In this and subsequent proofs, the subscript Δ is deleted from (w_Δ, π_Δ) when it is not needed, and w_Δ is suppressed from the list of arguments of p.

(i) We have

$$1 \geq \pi[0, x_n) + \pi[x_{n+1}, y_n) + \pi[y_n, B] > \pi[0, x_n) + 2\pi[y_n, B] - \frac{\rho}{K},$$

where the last inequality follows from the hypothesis. The inequality between the first and third terms gives the conclusion.

(ii) First consider the outflow from $[0, x_n)$. Note that producers with $x \geq x_n$ do not contribute to the outflow. Consider producers with $x < x_n$ and consumers with $m \geq y_n$. We have

$$x_n \leq \min p(z_n, 0) \leq \min p(y_n, 0) \leq x + \min p(y_n, x) \leq x + \min p(m, x),$$

where the first inequality follows from the definition of z_n, the second from the definition of y_n and Lemma 5(ii), the third from Lemma 5(iii), and the fourth from Lemma 5(ii). So a lower bound on the outflow is $\frac{1}{N}\pi[0, x_n)\pi[y_n, B]$.

Next consider the inflow into $[0, x_n)$. Note that consumers with $x < x_n$ do not contribute to the inflow. We start with consumers with $x \geq y_n$ and producers with $m \geq 0$. Apply Lemma 5(iv) with $(x', m') = (x_n, x_{n+1})$. (By the definition of $\{x_n\}$, $p(x_n, x_{n+1}) = \{0\}$. By the definition of y_n, $y_n \geq x_n + x_{n+1}$ and hence $m \geq 0 \geq m' - (x - x')$.) It follows that $\max p(x, m) \leq x - x' = x - x_n$ or $x - \max p(x, m) \geq x_n$. That is, consumers with $x \geq y_n$ do not contribute to the inflow. Now we consider consumers with $x \geq x_n$ and producers with $m \geq x_{n+1}$. Apply Lemma 5(iv) with $(x', m') = (x_n, x_{n+1})$. It follows that $x - \max p(x, m) \geq x_n$. That is, consumers with $x \geq x_n$ do not contribute to the inflow if they meet producers with $m \geq x_{n+1}$. So an upper bound on the inflow is $\frac{1}{N}\pi[x_n, y_n)\pi[0, x_{n+1})$.

Because (w, π) is a steady state, the outflow from and inflow into $[0, x_n)$ are equal. Therefore,

$$\pi[0, x_n)\pi[y_n, B] \leq \pi[x_n, y_n)\pi[0, x_{n+1}). \tag{15}$$

By the hypothesis, $\pi[y_n, B] > \pi[x_{n+1}, y_n)$. So (15) implies $x_{n+1} > x_n$. Now write $\pi[x_n, y_n)$ as $\pi[x_n, x_{n+1}) + \pi[x_{n+1}, y_n)$ and $\pi[0, x_{n+1})$ as $\pi[0, x_n) + \pi[x_n, x_{n+1})$ and insert these into (15). Then, letting $x \equiv \pi[x_n, x_{n+1})$, (15) is equivalent to $0 \leq x^2 + s_1 x - s_2$, where $s_1 = \pi[0, x_n) + \pi[x_{n+1}, y_n) > \rho$ (by Lemma 6(iv)) and $s_2 = \pi[0, x_n)\{\pi[y_n, B] - \pi[x_{n+1}, y_n)\} > \rho^2/K$ (by

Lemma 6(iv) and the hypothesis). Then it follows from the definition of $\hat{g}$ that $\pi[x_n, x_{n+1}) \geq \hat{g}$. ■

The next lemma is an application of Lemma 7. It provides the ingredients for the induction argument used in the proof of Proposition 1.

Lemma 8 *Let* $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$. *Let* $\{\omega_l\}$, ρ, *and* K *be as given in Definition* 2.

(*i*) *If* ω_1 *exists, then* $\pi_\Delta[\omega_1, B] < \frac{1}{2}[1 - \frac{K-1}{K}\rho]$.

(*ii*) *If* ω_l *exists and* $\pi_\Delta[\omega_{l-1}, B] < \frac{1}{2^{l-1}} - \frac{1}{2^{l-1}}\frac{K-\sum_{j=1}^{l-1} 2^{j-1}}{K}\rho$, *then* $\pi_\Delta[\omega_l, B] < \frac{1}{2^l} - \frac{1}{2^l}\frac{K-\sum_{j=1}^{l} 2^{j-1}}{K}\rho$.

Proof. (i) By definition, $\omega_1 = y_{\sigma(1)}$, and $y_{\sigma(l)}$ is determined by $\{(x_n, y_n, z_n)\}$ for $x_0 = \Delta$. If $\pi[y_n, B] - \pi[x_{n+1}, y_n) < \frac{\rho}{K}$ for some $1 \leq n \leq \sigma(1)$, then

$$\pi[\omega_1, B] \leq \pi[y_n, B] < \frac{1}{2}\{1 - \pi[0, x_n) + \frac{\rho}{K}\} < \frac{1}{2}[1 - \frac{K-1}{K}\rho],$$

where the first inequality follows from Lemma 6(iii), the second from Lemma 7(i), and the last from Lemma 6(iv). If $\pi[y_n, B] - \pi[x_{n+1}, y_n) \geq \frac{\rho}{K}$ for all $1 \leq n \leq \sigma(1)$, then by Lemma 7(ii), we have $x_{n+1} > x_n$ and $\pi[x_n, x_{n+1}) \geq \hat{g}$ for all $1 \leq n \leq \sigma(1)$. It follows that

$$\pi[0, \omega_1) \geq \pi[0, x_1) + \textstyle\sum_{n=1}^{\sigma(1)} \pi[x_n, x_{n+1}) > \rho + \sigma(1)\hat{g} \geq \displaystyle\frac{1}{2} + \rho,$$

where the second inequality follows from Lemma 6(iv) and the last from the definition of $\sigma(1)$. Hence $\pi[\omega_1, B] = 1 - \pi[0, \omega_1) < \frac{1}{2} - \rho$.

(ii) By definition, $\omega_l = y_{\sigma(l)}$, where $y_{\sigma(l)}$ is determined by $\{(x_n, y_n, z_n)\}$ for $x_0 = \omega_{l-1}$. If $\pi[y_n, B] - \pi[x_{n+1}, y_n) < \frac{\rho}{K}$ for some $1 \leq n \leq \sigma(l)$, then

$$\begin{aligned}
\pi[\omega_l, B] &\leq \pi[y_n, B] \\
&< \frac{1}{2}\{1 - \pi[0, x_n) + \frac{\rho}{K}\} \\
&\leq \frac{1}{2}\{\pi[x_0, B] + \frac{\rho}{K}\} \\
&< \frac{1}{2^l} - \frac{1}{2^l}\frac{K - \sum_{j=1}^{l} 2^{j-1}}{K}\rho,
\end{aligned}$$

where the first inequality follows from Lemma 6(iii), the second from Lemma 7(i), the third from $\pi[0, x_n) + \pi[x_0, B] \geq 1$, and the last from the hypothesis. If $\pi[y_n, B] - \pi[x_{n+1}, y_n) \geq \frac{\rho}{K}$ for all $1 \leq n \leq \sigma(l)$, then by Lemma 7(ii), we have $x_{n+1} > x_n$ and $\pi[x_n, x_{n+1}) \geq \hat{g}$ for all $1 \leq n \leq \sigma(l)$. Hence,

$$\begin{aligned}
\pi[0, \omega_l) &\geq \pi[0, x_1) + \textstyle\sum_{n=1}^{\sigma(l)} \pi[x_n, x_{n+1}) \\
&\geq 1 - \pi[x_0, B] + \sigma(l)\hat{g} \\
&> 1 - \frac{1}{2^{l-1}} + \frac{1}{2^{l-1}} \frac{K - \sum_{j=1}^{l-1} 2^{j-1}}{K} \rho + \frac{1}{2^l} \\
&= 1 - \frac{1}{2^l} + \frac{1}{2^{l-1}} \frac{K - \sum_{j=1}^{l-1} 2^{j-1}}{K} \rho \\
&> 1 - \frac{1}{2^l} + \frac{1}{2^l} \frac{K - \sum_{j=1}^{l} 2^{j-1}}{K} \rho,
\end{aligned}$$

where the third inequality follows from the hypothesis and the definition of $\sigma(l)$. The conclusion follows from $\pi[\omega_l, B] = 1 - \pi[0, \omega_l)$. ■

By Lemma 8, if the sequence $\{\omega_l\}_{l=1}^{L}$ exists and if ω_L is sufficiently small, then we get a contradiction to the assumption that the mean of money holdings is unity. The rest of this section shows that if $w_\Delta(\Delta)/\Delta$ is unbounded as Δ approaches 0, then there exists $\{\omega_l\}_{l=1}^{L}$ satisfying those conditions. As a first step, the next lemma shows that large $w_\Delta(x_0, \Delta)/\Delta$ implies small x_n and gives a sufficient condition for existence of z_n and y_n.

Lemma 9 *Let $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$. Let $\{(x_n, y_n, z_n)\}$ for $x_0 \in B_\Delta \backslash \{0\}$ be as given in Definition 2.*

(i) $x_n < W[u'(0)]^n [w_\Delta(x_0, \Delta)/\Delta]^{-1}$.

(ii) For any n, if $w_\Delta(x_0, \Delta)/\Delta$ is sufficiently large, then z_n and y_n exist.

Proof. See the appendix. ■

The next lemma is crucial for existence of $\{\omega_l\}_{l=1}^{L}$.

Lemma 10 *Let $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$. Let $\{(x_n, y_n, z_n)\}$ for $x_0 \in B_\Delta \backslash \{0\}$ and $\{(x_n^*, y_n^*, z_n^*)\}$ for $x_0^* = \Delta$ be as given in Definition 2. If y_n and $y_{\sigma(1)}^*$ exist and $8x_1^* \leq B$, then there exist $c_n > 0$ and $C > 0$, not dependant on Δ, such that $w_\Delta(y_n, \Delta) > \min\{c_n w_\Delta(x_0, \Delta), C w_\Delta(\Delta)\}$.*

Proof. See the appendix. ■

The next lemma gives the desired sufficient condition for existence of $\{\omega_l\}_{l=1}^L$.

Lemma 11 *Let $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$. Let $\{\omega_l\}$ and L be as given in Definition 2. If $w_\Delta(\Delta)/\Delta$ is sufficiently large, then $\{\omega_l\}_{l=1}^L$ exists and $w_\Delta(\omega_l, \Delta) > \xi_l w_\Delta(\Delta)$, where $\xi_l > 0$ does not depend on Δ.*

Proof. First, we consider $l = 1$. By definition, $\omega_1 = y_{\sigma(1)}$, and $y_{\sigma(l)}$ is determined by $\{(x_n, y_n, z_n)\}$ for $x_0 = \Delta$. Note that this $\{(x_n, y_n, z_n)\}$ sequence is the $\{(x_n^*, y_n^*, z_n^*)\}$ sequence in Lemma 10. Lemma 9(ii) and sufficiently large $w(\Delta)/\Delta$ imply that $y_{\sigma(1)}$ exists and $8x_1 \leq B$. Now apply Lemma 10 for $n = \sigma(1)$. We have $w(\omega_1, \Delta) > \min\{c_{\sigma(1)}, C\}w(\Delta)$. (Note that $w(\Delta, \Delta) = w(\Delta)$.) Then we set $\xi_1 = \min\{c_{\sigma(1)}, C\}$. Given the result for $l = 1$, it suffices to prove that if $w(\omega_{l-1}, \Delta) > \xi_{l-1}w(\Delta)$, where $\xi_{l-1} > 0$ does not depend on Δ, then ω_l exists and $w(\omega_l, \Delta) > \xi_l w(\Delta)$, where $\xi_l > 0$ does not depend on Δ. By definition, $\omega_l = y_{\sigma(l)}$, and $y_{\sigma(l)}$ is determined by $\{(x_n, y_n, z_n)\}$ for $x_0 = \omega_{l-1}$. Lemma 9(ii) and sufficiently large $w(\Delta)/\Delta$ and $w(\omega_{l-1}, \Delta) > \xi_{l-1}w(\Delta)$ imply that ω_l exists. Now apply Lemma 10 for $n = \sigma(l)$. We have $w(\omega_l, \Delta) > \min\{c_{\sigma(l)}, Cw(\Delta)\}w(\Delta)$. (Note that $w(\omega_{l-1}, \Delta) > \xi_{l-1}w(\Delta)$.) Then we set $\xi_l = \min\{c_{\sigma(l)}\xi_{l-1}, C\}$. ■

I can now prove that $w_\Delta(\Delta)/\Delta$ is uniformly bounded.

Proposition 1 *Let $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$. There exists M, not dependent on Δ, such that $w_\Delta(\Delta)/\Delta < M$.*

Proof. Assume by contradiction that $w_\Delta(\Delta)/\Delta$ is unbounded as Δ goes to 0. Then by Lemma 11, for (w_Δ, π_Δ) with sufficiently large $w_\Delta(\Delta)/\Delta$, the sequence $\{\omega_l\}_{l=1}^L$ given in Definition 2 exists. By Lemma 8(i), $\pi_\Delta[\omega_1, B] < \frac{1}{2}[1 - \frac{K-1}{K}\rho]$. By Lemma 8(ii) and by induction,

$$\pi_\Delta[\omega_L, B] < \frac{1}{2^L} - \frac{1}{2^L}\frac{K - \sum_{j=1}^L 2^{j-1}}{K}\rho = \frac{1}{2^L} \leq \frac{1}{2}\mu[\hat{x}, B], \tag{16}$$

where the equality and the last inequality follow from the definitions of L and K. Now we have $W > w_\Delta(\omega_L) \geq \omega_L w_\Delta(\omega_L, \Delta)/\Delta > \omega_L \xi_L w_\Delta(\Delta)/\Delta$, where the third inequality follows from Lemma 11. It follows that $\omega_L < 1$ for sufficiently large $w_\Delta(\Delta)/\Delta$. Because $\hat{x} \geq 1$ and μ is continuous at $\hat{x}$, $\pi_\Delta[\omega_L, B] \geq \pi_\Delta[\hat{x}, B] = \mu_\Delta[\hat{x}, B] > \frac{1}{2}\mu[\hat{x}, B]$ for small Δ. But this contradicts (16). ■

5 Completion of the proof

I begin by proving all of Theorem 1 except for full support of μ.

Proposition 2 *If $B \geq 8$ and $\infty > u'(0) \geq [4/(R\beta)]^2$, then there exists a monetary steady state, (v, μ), where v is continuous, strictly increasing, and strictly concave.*

Proof. Let (v, μ) be a Lemma 3 limit point. In this and subsequent proofs, we suppress v from the list of arguments of $\tilde{f}$ and $\tilde{p}$. By Proposition 1, $v(x) < Mx$, and hence v is continuous at 0. By Lemma 3, v is concave and strictly increasing: by concavity, v is continuous on $(0, B)$ (see [1, 5.29 Theorem, p. 179]); by concavity and monotonicity, v is continuous at B. Hence $v \in \mathbf{V}$. Then by Lemma 4, (v, μ) is a monetary steady state. For strict concavity of v, let $0 \leq x_1 < x_2 \leq B$, $0 < \alpha < 1$, and $x = \alpha x_1 + (1-\alpha)x_2$. Because v is a steady state value function and $v(x_2) > 0$, there exists some $A \subset [0, B]$ with $\mu(A) > 0$ and $\tilde{p}(x_2, m) > 0$ for $m \in A$. But then $\alpha\tilde{p}(x_1, m) + (1-\alpha)\tilde{p}(x_2, m) \leq x$ implies $\tilde{f}(x, m) > \alpha\tilde{f}(x_1, m) + (1-\alpha)\tilde{f}(x_2, m)$. This gives strict concavity of v. ∎

In what follows, let (v, μ) be a Proposition 2 steady state and let *supp* μ denote the support of μ. (For the definition and existence of the support of a measure, see [1, p. 374].) The rest of this section shows that μ has full support. As in [11], the full-support property comes from the properties of the value function. Although many of the ideas in [11] are used here, the arguments are not identical. For an indivisible-money steady state (w, π) with Δ as the unit of money, by a straightforward induction argument, $\pi(\Delta) > 0$ implies full support, while $\pi(\Delta) = 0$ implies a periodic support. The periodic support is ruled out by another argument. For the (v, μ) steady state, an analogue to $\pi(\Delta) > 0$ is $\inf\{x : \mu(0, x) > 0\} = 0$. Indeed, $\inf\{x : \mu(0, x) > 0\} = 0$ implies full support, but with quite a different argument. Moreover, $\inf\{x : \mu(0, x) > 0\} > 0$ may or may not imply a periodic support. The structure of the proof is as follows. Lemma 13 shows that the bound on money holdings is binding; then Lemma 14 shows that $\inf\{x : \mu(0, x) > 0\} = 0$ implies full support; then Proposition 3 uses that sufficient condition to establish full support.

The next lemma collects some preliminary results, mainly the dependence of the optimal offer in (2) on the money holdings of the consumer and the producer.

Lemma 12 *(i) If* $\tilde{p}(x_1, m_1, v) = 0$*,* $x_2 \geq x_1$*, and* $m_2 \geq m_1 - (x_2 - x_1)$*, then* $\tilde{p}(x_2, m_2, v) \leq x_2 - x_1$*.*

(ii) If $m_1 < m_2$*, then* $m_1 + \tilde{p}(x, m_1, v) \leq m_2 + \tilde{p}(x, m_2, v)$*.*

(iii) If $x_2 > x_1$*, then* $\tilde{p}(x_2, m, v) - \tilde{p}(x_1, m, v) \in [0, x_2 - x_1]$*.*

(iv) If $x > m$*, then* $\tilde{p}(x, m, v) > 0$*.*

(v) Let $x, m \in$ *supp* μ *and let* Q *be an open set in* $[0, B]$*. If either* $x - \tilde{p}(x, m, v) \in Q$ *or* $x + \tilde{p}(m, x, v) \in Q$*, then* $\mu Q > 0$*.*

Proof. See the appendix. ∎

The next lemma shows that there is no endogenous bound on money holdings.

Lemma 13 $\mu[0, x] = 1$ *if and only if* $x = B$*.*

Proof. See the appendix. ∎

The next lemma shows that $\inf\{x : \mu(0, x) > 0\} = 0$ implies full support.

Lemma 14 *If* $\inf\{x : \mu(0, x) > 0\} = 0$*, then supp* $\mu = [0, B]$*.*

Proof. Assume by contradiction that there exist $z^*, y^* \in$ *supp* μ with $\mu(z^*, y^*) = 0$. By the hypothesis, $z^* > 0$; otherwise $\mu(z^*, y^*) > 0$. Let $\varepsilon^* = \frac{y^* - z^*}{2y^*}$. We use the following claim to draw a contradiction. *Claim: If there exist* $z_i, y_i \in$ *supp* μ *with* $\mu(z_i, y_i) = 0$*, then there exist* $z_{i+1}, y_{i+1} \in$ *supp* μ *with* $\mu(z_{i+1}, y_{i+1}) = 0$ *and* $y_{i+1} \leq z_i$ *and* $y_{i+1} - z_{i+1} > (y_i - z_i)(1 - 2\varepsilon^*)$*.* With the claim, by setting $z_1 = z^*$ and $y_1 = y^*$, and by induction, we have $\sum_{i=1}^{n}(y_i - z_i) > (y^* - z^*) \sum_{i=1}^{n}(1 - 2\varepsilon^*)^{i-1} > y^*$ for large n, a contradiction.

Now we give a proof of the claim. It suffices to consider $i = 1$. Let z_1 and y_1 satisfy the hypothesis of the claim, and let $c = y_1 - z_1$. Fix positive $\varepsilon < \min\{\varepsilon^*, \frac{z_1}{c}\}$. Note that $\varepsilon < 1$ and $c\varepsilon < z_1$.

First assume $\tilde{p}(z_1, z_1) = 0$. Let $A_{z_1} = (\max\{0, z_1 - c + c\varepsilon\}, z_1 - c\varepsilon)$. Assume $\mu A_{z_1} > 0$ and let $z \in A_{z_1} \cap$ *supp* μ. By Lemma 12(i), $\tilde{p}(z_1, z_1) = 0$ implies $\tilde{p}(y_1, z) \leq c$. Either $\tilde{p}(y_1, z) = c$ or $\tilde{p}(y_1, z) < c$. If the latter, then $y_1 - \tilde{p}(y_1, z) > z_1$. By Lemma 12(iv), $y_1 > z$ implies $\tilde{p}(y_1, z) > 0$. It follows that $y_1 - \tilde{p}(y_1, z) \in (z_1, y_1)$. Because $y_1, z \in$ *supp* μ, by Lemma 12(v), this implies $\mu(z_1, y_1) > 0$, a contradiction. So $\tilde{p}(y_1, z) = c$. It follows that $z + \tilde{p}(y_1, z) \in (z_1, y_1)$. Then by Lemma 12(v), $\mu(z_1, y_1) > 0$, a contradiction. So it must be $\mu A_{z_1} = 0$. This and the hypothesis of the lemma imply

$z_1 - c + c\varepsilon > 0$; otherwise, $A_{z_1} = (0, z_1 - c\varepsilon)$, and this implies $\mu A_{z_1} > 0$. By $\mu A_{z_1} = 0$, there exist $z_2, y_2 \in$ *supp* μ such that $z_2 \le z_1 - c + c\varepsilon$ and $z_1 - c\varepsilon \le y_2 \le z_1$ and $\mu(z_2, y_2) = 0$.

Next assume $\tilde{p}(z_1, z_1) > 0$. Let $\underline{z} = \max\{x : \tilde{p}(x, z_1) = 0\}$. Either $z_1 - \underline{z} \ge c$ or $z_1 - \underline{z} < c$. If the latter, then by Lemma 12(i), $\tilde{p}(z_1, z_1) < c$. Because $\tilde{p}(z_1, z_1) > 0$, it follows that $z_1 + \tilde{p}(z_1, z_1) \in (z_1, y_1)$. Then by Lemma 12(v), $\mu(z_1, y_1) > 0$, a contradiction. So $z_1 - \underline{z} \ge c$. Let $A_{\underline{z}} = (\underline{z} + c\varepsilon, \underline{z} + c - c\varepsilon)$. Assume $\mu A_{\underline{z}} > 0$ and let $z \in A_{\underline{z}} \cap$ *supp* μ. By Lemma 12(i), $\tilde{p}(\underline{z}, z_1) = 0$ implies $\tilde{p}(z, z_1) \le z - \underline{z} < c - 2c\varepsilon$. Hence $z_1 + \tilde{p}(z, z_1) < y_1$. By the definition of $\underline{z}$, $z > \underline{z}$ implies $\tilde{p}(z, z_1) > 0$. It follows that $z_1 + \tilde{p}(z, z_1) \in (z_1, y_1)$. Then by Lemma 12(v), $\mu(z_1, y_1) > 0$, a contradiction. So it must be $\mu A_{\underline{z}} = 0$. By $\mu A_{\underline{z}} = 0$, there exist $z_2, y_2 \in$ *supp* μ such that $z_2 \le \underline{z} + c\varepsilon$ and $\underline{z} + c - c\varepsilon \le y_2 \le z_1$ and $\mu(z_2, y_2) = 0$. This establishes the claim. ■

Now I can prove the following proposition.

Proposition 3 *If (v, μ) is a Proposition 2 steady state, then μ has full support.*

Proof. Let $a \equiv \inf\{x : \mu(0, x) > 0\}$. By an argument in [11], *supp* $\mu \neq \{0, B\}$.[3] So a is well defined. By Lemma 14, it suffices to show $a = 0$. Assume by contradiction that $a > 0$. Note that $a \in$ *supp* μ.

First, we claim that $\mu\{0\} > 0$. Suppose otherwise. It follows that $\mu[a, B] = 1$. Now, either $\tilde{p}(a, a) > 0$ or $\tilde{p}(a, a) = 0$. If the latter, then by Lemma 12(i), $\tilde{p}(a, m) = 0$ for $m \ge a$. But because $\mu[a, B] = 1$, this implies $v\{a\} = 0$, a contradiction. So $\tilde{p}(a, a) > 0$ or $a - \tilde{p}(a, a) < a$. Because $a \in$ *supp* μ, by Lemma 12(v), this implies $\mu[0, a) > 0$, a contradiction.

Next, we consider two exhaustive cases for $\mu\{a\}$: the first has an indivisible-money counterpart, while the second does not.

Case 1 : $\mu\{a\} > 0$. (Here, both 0 and a are mass points and there is no mass between them.) By the argument in the proof of [11, Lemma 8], this implies *supp* $\mu = \{0, a, 2a, ..., B\}$. As we show in the proof of [11, Lemma 9], this implies that there exists a mapping θ from a set with at least three positive vectors in $\mathbb{R}_{B/a}$ to $\mathbb{R}_{B/a}$ with the following properties: θ is concave and strictly increasing with $\theta(0) \ge 0$ and has multiple positive fixed points. That, however, is impossible.

[3] The argument is in the proof of Claim 1 of [11, Lemma 8]. In that proof, $a < B$ is taken implicitly: $a = B \Rightarrow \rho > 1 - 1/B$, but this and (18) lead to an obvious contradiction. This is the needed argument.

Case 2 : $\mu\{a\} = 0$. (Here, 0 is a mass point but a is not and there is no mass between them.) It follows that $(a, a+\varepsilon) \cap \mathit{supp}\ \mu$ is non empty for $\varepsilon > 0$. Lemma 12(v) and $\mu(0, a) = 0$ imply $\tilde{p}(a, a) \in \{0, a\}$. But by the argument in Claim 1 in the proof of [11, Lemma 8], $\tilde{p}(a, a) = 0$ implies a contradiction. So $\tilde{p}(a, a) = a$. Then by continuity of $\tilde{p}$, $\tilde{p}(a, m) > 0$ for some $m > a$. Because $(a, a+\varepsilon) \cap \mathit{supp}\ \mu$ is non empty for $\varepsilon > 0$, $A = \mathit{supp}\ \mu \cap \{m > a : \tilde{p}(a, m) > 0\}$ is non empty. Let $b = \sup A$. (Note that $b \in \mathit{supp}\ \mu$.) Now by $\mu(0, a) = 0$ and Lemma 12(v), $\tilde{p}(a, m) \in \{0, a\}$ for $m \in \mathit{supp}\ \mu$, and hence $\tilde{p}(a, m) = a$ for $m \in A$. By continuity of $\tilde{p}$, this implies $\tilde{p}(a, b) = a$. Then by Lemma 12(iii), $\tilde{p}(x, b) \geq a$ for $x > a$. Now fix positive $\varepsilon < a$ with $a + \varepsilon \in \mathit{supp}\ \mu$ and fix positive $\delta < \varepsilon/2$. By $\tilde{p}(a+\varepsilon, b) \geq a > \varepsilon$, $\mu(0, a) = 0$, and Lemma 12(v), $\tilde{p}(a+\varepsilon, b) = a + \varepsilon$. This and Lemma 12(iii) imply $\tilde{p}(x, b) = x$ for $x \in (a, a+\delta)$. Then, by Lemma 12(v), $(b+a, b+a+\delta)$ $\cap\ \mathit{supp}\ \mu$ is non empty because $(a, a+\delta) \cap \mathit{supp}\ \mu$ is non empty. Therefore, $\tilde{p}(a, m) > 0$ for $m \in (b+a, b+a+\delta)$ implies sup $A > b$, a contradiction. So fix $m \in (b+a, b+a+\delta)$ and let $p = \tilde{p}(\varepsilon/2, b+a+\varepsilon/2)$. Now $p = 0$ and Lemma 12(i) imply $\tilde{p}(a+\varepsilon, b) \leq a+\varepsilon/2$. But that is impossible because $\tilde{p}(a+\varepsilon, b) = a+\varepsilon$. So $p > 0$, and then by Lemma 12(iii), $\tilde{p}(a, b+a+\varepsilon/2) > 0$. But this and $\delta < \varepsilon/2$ and Lemma 12(i) imply $\tilde{p}(a, m) > 0$, the desired contradiction. ■

6 The concluding remarks

It has been shown that in the model studied, there exists a monetary steady state with a strictly increasing and strictly concave value function and with full support, a nice steady state. There are a number of restrictive assumptions in the model. Some are inherited from the indivisible-money model studied in [11]. There is an arbitrary bound on individual holdings and there are take-it-or-leave-it offers by consumers. As in [11], the bound assures that the set of measures is compact and take-it-or-leave-it offers assure that the mapping studied preserves concavity of value functions. A new assumption is that $u'(0)$ is finite. It is used to get continuity of a limit of embedded (indivisible-money) steady-state value functions. In particular, it implies that the slope of the embedded value functions is uniformly bounded. Indeed, it is necessary for such boundedness. However, I doubt that finiteness of $u'(0)$ is necessary for continuity of the limit value function.

There are, of course, additional questions one could ask about nice steady states. One is about local stability. A steady state (v, μ) is locally stable if for any μ_0 close enough to μ there exists an equilibrium that approaches (v, μ). Is at least one nice steady state locally stable? Another is about uniqueness. As discussed in footnote 2, the model has a multiplicity of steady states. But, are there multiple nice steady states? Finally, what can be said about steady-state distributions of money holdings? For example, can it be shown that the "density function" has a single peak? And, can it be shown that the distribution does not have a mass point?

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Appendix

To complete the proofs that are relegated to the appendix, I need some intermediate results. Those results are given below as Lemmas 15, 16, and 17. Lemma 15 is used in the proof of Lemmas 3 and 10. Lemma 16 is used in the proof of Lemmas 9 and 10. Lemma 17, using Lemma 15 in its proof, is the key intermediate step in the proof of Lemma 10.

Lemma 15 *Let* (w_Δ, π_Δ) *be a Lemma* 2 *steady state.*

(i)If $b_0, b_1, b_2 \in B_\Delta$ *is such that* $b_0 < b_1 \leq b_2$ *and* $b_0 + b_2 \leq 2b_1$*, then* $w_\Delta(b_2, \Delta) > R\pi_\Delta[0, b_0]\beta u'(W) w_\Delta(b_1, \Delta)$.

(ii) If $b_1, b_2 \in B_\Delta$ *is such that* $b_1 \leq b_2$, $0 < b_2 \leq B/2$, *and* $\pi_\Delta[b_1, b_2] \geq 1/4$, *then* $w_\Delta(2b_2) - w_\Delta(b_1) \geq D/\beta$.

Proof. In this and subsequent proofs, the subscript Δ is deleted from (w_Δ, π_Δ) when it is not needed, and w_Δ is suppressed from the list of arguments of f and p. The proof of the lemma follows the exact logic used in the proof of [11, Lemma 3].

(i) Fix $m \in [0, b_0]$. If $p(b_2 - \Delta, m) \ni p \geq b_2 - b_1$, then by $p \in \Gamma_\Delta(b_2, m)$, $f(b_2, m) - f(b_2 - \Delta, m) \geq \beta w(b_2 - p, \Delta) \geq \beta w(b_1, \Delta)$. If $p(b_2 - \Delta, m) \ni p < b_2 - b_1$, then by $p + \Delta \in \Gamma_\Delta(b_2, m)$,

$$\begin{aligned} f(b_2, m) - f(b_2 - \Delta, m) &\geq u[\beta w(m + p + \Delta, p + \Delta)] - u[\beta w(m + p, p)] \\ &> u'[\beta w(m + p + \Delta, p + \Delta)]\beta w(m + p + \Delta, \Delta) \\ &\geq u'[\beta w(b_1)]\beta w(b_1, \Delta). \end{aligned}$$

Therefore, by the definition of W, $f(b_2, m) - f(b_2 - \Delta, m) > \beta u'(W) w(b_1, \Delta)$. Because (w, π) is a steady state, this implies the result.

(ii) Assume by contradiction that $w(2b_2) - w(b_1) < D/\beta$. It follows that $w(b_1 + b_2, b_2) = w(b_1 + b_2) - w(b_1) < D/\beta$. Fix $m \in [b_1, b_2]$. The logic used here is similar to that used in the proof of part (i). If $p(2b_2 - \Delta, m) \ni p \geq b_2$, then $f(2b_2, m) - f(2b_2 - \Delta, m) \geq \beta w(b_2, \Delta)$. If $p(b_2 - \Delta, m) \ni p < b_2$, then

$$\begin{aligned} f(2b_2, m) - f(2b_2 - \Delta, m) &> u'[\beta w(m + p + \Delta, p + \Delta)]\beta w(m + p + \Delta, \Delta) \\ &\geq u'[\beta w(b_1 + b_2, b_2)]\beta w(2b_2, \Delta) \\ &> \beta u'(D) w(2b_2, \Delta). \end{aligned}$$

Now if $w(b_2, \Delta) \geq u'(D) w(2b_2, \Delta)$, then

$$w(2b_2, \Delta) > R\pi[b_1, b_2]\beta u'(D) w(2b_2, \Delta) \geq (R\beta/4) u'(D) w(2b_2, \Delta) > w(2b_2, \Delta),$$

a contradiction. So it must be that $w(b_2, \Delta) < u'(D) w(2b_2, \Delta)$. Then

$$w(2b_2, \Delta) > R\pi[b_1, b_2]\beta w(b_2, \Delta) \geq (R\beta/4) w(b_2, \Delta).$$

Again fix $m \in [b_1, b_2]$. Because $p(b_2 - \Delta, m) + \Delta \subset \Gamma_\Delta(b_2, m)$, by the logic used above, $f(b_2, m) - f(b_2 - \Delta, m) > \beta u'(D) w(2b_2, \Delta)$. It follows that

$$w(b_2, \Delta) > R\pi[b_1, b_2]\beta u'(D) w(2b_2, \Delta) > (R\beta/4)^2 u'(D) w(b_2, \Delta) > w(b_2, \Delta),$$

a contradiction. ■

Proof of Strict Monotonicity of v in Lemma 3

Proof. Assume by contradiction that v is not strictly increasing. Because v is concave and non decreasing, there exists a unique $x_2 > 0$ with $v(x) = v(x_2)$ for $x \geq x_2$ and $v(x) < v(x_2)$ for $x < x_2$. Let x_0 be the unique solution for $v(x_2) - v(x_0) = D/2$. Recall by assumption, $\Delta = B10^{-n}$ for some $n \in \mathbb{N}$. Without loss of generality, assume $B \in \mathbb{N}$. Then by taking large n and taking some arbitrarily close approximation, we can assume that $x_0 10^n$, $x_2 10^n \in \mathbb{N}$, and, hence, $x_0, x_2 \in B_\Delta$ for small Δ. The approximation, if taken, can satisfy the following essential requirements for x_0 and x_2: $v(x) = v(x_2)$ for $x \geq x_2$, $v(x_2) - v(x_0) < D$, and $v(x_2) - v(x_1) > 0$, where x_1 is defined as follows. If $x_2 > 4$, then let $x_1 = x_2 - \min\{x_2 - 2, B - x_2\}/10$. If $x_2 \leq 4$, then let $x_1 = x_2 - \min\{x_2 - x_0, B - x_2\}/10$. Also, let $x_3 = (x_2 - x_1) + x_2$. Note that $x_0, x_2 \in B_\Delta \Rightarrow x_1, x_3 \in B_\Delta$ for small Δ.

Let $\varepsilon > 0$ satisfy $[v(x_2) - v(x_1) - \varepsilon]/\varepsilon > [R\beta u'(W)/4]^{-1}$. For small Δ, we have

$$\begin{aligned}\frac{w_\Delta(x_1,\Delta)}{w_\Delta(x_3,\Delta)} &\geq \frac{[w_\Delta(x_2) - w_\Delta(x_1)]/(x_2 - x_1)}{[w_\Delta(x_3) - w_\Delta(x_2)]/(x_3 - x_2)} \\ &\geq \frac{v(x_2) - v(x_1) - \varepsilon}{v(x_3) - v(x_2) + \varepsilon} \\ &= [v(x_2) - v(x_1) - \varepsilon]/\varepsilon \\ &> [R\beta u'(W)/4]^{-1},\end{aligned} \tag{17}$$

where the first inequality follows from concavity of w_Δ, the second from $x_2 - x_1 = x_3 - x_2$ and $w_\Delta(x_i) = v_\Delta(x_i)$ and $\lim v_\Delta(x_i) = v(x_i)$ for $i = 1, 2, 3$, and the equality from $v(x_3) = v(x_2)$.

Now we discuss the two possible cases for x_2.

(i) $x_2 > 4$. Let (w_Δ, π_Δ) satisfy (17) and apply Lemma 15(i) with $b_0 = 2$, $b_1 = x_1$, and $b_2 = x_3$. (Note that $2 + x_3 - x_1 = 2 + 2(x_2 - x_1) \leq 2 + 2 \times \frac{x_2 - 2}{10} < x_2 - \frac{x_2 - 2}{10} \leq x_1$.) It follows that $w_\Delta(x_3, \Delta) > R\pi_\Delta[0, 2]\beta u'(W) w_\Delta(x_1, \Delta) > [R\beta u'(W)/4] w_\Delta(x_1, \Delta)$, which contradicts (17).

(ii) $x_2 \leq 4$. It follows from $v(8) - v(x_0) = v(x_2) - v(x_0) < D$ and $\lim v_\Delta(x) = v(x)$ for $x = 8, x_0$ that for small Δ,

$$v_\Delta(8) - v_\Delta(x_0) < D. \tag{18}$$

Let (w_Δ, π_Δ) satisfy (17) and (18). Either $\pi_\Delta[x_0, 8] \leq 1/2$ or $\pi_\Delta[x_0, 8] > 1/2$. If the latter, then by $\pi_\Delta(4, 8] < 1/4$, $\pi_\Delta[x_0, 4] > 1/4$. This and Lemma 15(ii) imply $v_\Delta(8) - v_\Delta(x_0) = w_\Delta(8) - w_\Delta(x_0) \geq D/\beta$, which contradicts

(18). So $\pi_\Delta[x_0, 8] \le 1/2$. Then by $\pi_\Delta[0,8] > 7/8$, $\pi_\Delta[0, x_0) > 3/8$. Now we apply Lemma 15(i) with $b_0 = x_0$, $b_1 = x_1$, and $b_2 = x_3$. (Note that $x_0 + x_3 - x_1 = x_0 + 2(x_2 - x_1) \le x_0 + 2 \times \frac{x_2 - x_0}{10} < x_2 - \frac{x_2 - x_0}{10} \le x_1$.) It follows that $w_\Delta(x_3, \Delta) > R\pi_\Delta[0, x_0]\beta u'(W) w_\Delta(x_1, \Delta) > [R\beta u'(W)/4] w_\Delta(x_1, \Delta)$, which contradicts to (17). ■

Proof of Lemma 6

Proof. By Lemma 5(i), $p(x_{n-1}, x_{n-1} - \Delta) \neq \{0\}$. Hence $x_n \ge x_{n-1}$. This gives part (i). Part (i) and Lemma 5(ii) imply parts (ii). Then parts (i) and (ii) imply (iii). Now consider part (iv). Let $\{(x_n^*, y_n^*, z_n^*)\}$ for $x_0^* = \Delta$ be as given in Definition 2. Lemma 5(ii) and $x_0 \ge \Delta$ imply $x_1 \ge x_1^*$. By part (i), $x_n \ge x_1^*$. So it suffices to show $\pi[0, x_1^*) > \rho$. We have

$$\begin{aligned} w(\Delta) &= R\{\textstyle\sum_{m=0}^{x_1^* - \Delta} \pi(m) u[\beta w(m + \Delta, \Delta)] + \sum_{m \ge x_1^*} \pi(m) \beta w(\Delta)\} \qquad (19) \\ &< R\{\textstyle\sum_{m=0}^{x_1^* - \Delta} \pi(m) u'(0) \beta w(m + \Delta, \Delta) + \pi[x_1^*, B] \beta w(\Delta)\} \\ &\le R\pi[0, x_1^*) u'(0) \beta w(\Delta) + R\{1 - \pi[0, x_1^*)\} \beta w(\Delta), \end{aligned}$$

where the equality follows from the definition of x_1^*, the first inequality from strict concavity of u and $u(0) = 0$, and the second inequality from concavity of w. The desired result follows from (19) and the definition of ρ. ■

Lemma 16 *Let* $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$. *Let* $\{(x_n, y_n, z_n)\}$ *for* $x_0 \in B_\Delta \backslash \{0\}$ *be as given in Definition* 2.

(i) $w_\Delta(x_n, \Delta) > [u'(0)]^{-n} w_\Delta(x_0, \Delta)$.

(ii) If $z_n > x_n$, *then* $w_\Delta(z_n - x_n, \Delta) > u'(W) w_\Delta(x_n, \Delta)$.

Proof. By the definition of x_n, we have

$$\beta w(x_{n-1}, \Delta) \le u[\beta w(x_n, \Delta)] < \beta u'(0) w(x_n, \Delta) \le \beta u'(0) w(x_n) \Delta / x_n. \qquad (20)$$

(The last inequality in (20) uses concavity of w and $w(0) = 0$.) A comparison of the first term with the third term in (20) gives $w(x_n, \Delta) > [u'(0)]^{-1} w(x_{n-1}, \Delta)$. This implis part (i). By the definition of z_n, $\min p(z_n - \Delta, 0) \le x_n - \Delta$ and $\min p(z_n, 0) \ge x_n$. Then by Lemma 5(ii), $x_n - \Delta \in p(z_n - \Delta, 0)$. It follows that $\beta w(z_n - x_n, \Delta) \ge u[\beta w(x_n)] - u[\beta w(x_n - \Delta)] > \beta u'[\beta w(x_n)] w(x_n, \Delta)$. This gives part (ii). ■

Proof of Lemma 9

Proof. A comparison of the first and last terms in (20) and $w(x_n) < W$ give $x_n < W[u'(0)][w(x_{n-1}, \Delta)/\Delta]^{-1}$. This implies part (i). By part (i) and Lemma 16(i), if $w(x_0, \Delta)/\Delta$ is sufficiently large, then $x_n < 1$ and $7w(x_n, \Delta)/\Delta > W/u'(W)$. By concavity of w, $w(7, \Delta)/\Delta < W/7$. Hence $u'(W)w(x_n, \Delta) > w(7, \Delta)$. If z_n does not exist, then $\{x : \min p(x, 0) \geq x_n\}$ is empty. In paricular, in a meeting between a consumer with 8 and a producer with 0, the consumer's spending is no more than $x_n - \Delta$. Hence $\beta w(8-x_n+\Delta, \Delta) > u[\beta w(x_n)]-u[\beta w(x_n-\Delta)] > \beta u'[\beta w(x_n)]w(x_n, \Delta)$. This and $x_n < 1$ imply $w(7, \Delta) > u'(W)w(x_n, \Delta)$, a contradiction. So sufficiently large $w(x_0, \Delta)/\Delta$ implies existence of z_n, and, hence, existence of y_n. ■

Lemma 17 *Let $(w_\Delta, \pi_\Delta) \in \{(w_\Delta, \pi_\Delta)\}_\Delta$. Let $\{(x_n, y_n, z_n)\}$ for $x_0 = \Delta$ be as given in Definition 2. Assume that $y_{\sigma(1)}$ exists and $8x_1 \leq B$. Then there exists $C_0 > 0$, not dependant on Δ, such that $w_\Delta(2x_1, \Delta) > C_0 w_\Delta(\Delta)$.*

Proof. In this proof, we denote $\sigma(1)$ by J. We start with three claims.

Claim 0 For $n \geq 2$, $w_\Delta(n\Delta, \Delta) > (R\rho\beta)^{n-1} w_\Delta(\Delta)$.

Fix $m \in [0, x_1)$. By the definition of x_1, $\max p(\Delta, m) \geq \Delta$. This and Lemma 5(ii) imply $p(n\Delta - \Delta, m) \ni p_n \geq \Delta$ for $n \geq 2$. Hence $f(n\Delta, m) - f(n\Delta - \Delta, m) \geq \beta w(n\Delta - p_n, \Delta) \geq \beta w(n\Delta - \Delta, \Delta)$. By Lemma 6(iv), $\pi[0, x_1) > \rho$. It follows that $w(n\Delta, \Delta) > R\rho\beta w(n\Delta - \Delta, \Delta)$. This and $w(\Delta, \Delta) = w(\Delta)$ imply the result.

Claim 1 If $x_{J+1} < 2x_1$ and $z_J \geq 4x_1$, then $w_\Delta(2x_1, \Delta) > u'(W)[u'(0)]^{-J} w_\Delta(\Delta)$.

By Lemma 6(i), $x_J \leq x_{J+1}$. Hence $x_J < 2x_1$. It follows that $w(2x_1, \Delta) > w(z_J - x_J, \Delta) > u'(W)w(x_J, \Delta) > u'(W)[u'(0)]^{-J} w(\Delta, \Delta)$, where the second inequality follows from Lemma 16(ii) and the third from Lemma 16(i). This and $w(\Delta, \Delta) = w(\Delta)$ imply the result.

Claim 2 If $x_{J+1} < 2x_1$ and $z_J < 4x_1$, then there exists $\alpha > 0$, not dependent on Δ, such that $w_\Delta(2x_1, \Delta) > \alpha w_\Delta(\Delta)$.

With Claims 1 and 2, we can easily prove the lemma. If $x_{J+1} \geq 2x_1$, then $w(2x_1, \Delta) \geq w(x_{J+1}, \Delta) > [u'(0)]^{-(J+1)} w(\Delta)$, where the last inequality follows from Lemma 16(i). If $x_{J+1} < 2x_1$, then by Claims 1 and 2, $w(2x_1, \Delta) > \min\{u'(W)[u'(0)]^{-J}, \alpha\} w(\Delta)$. Hence the desired C_0 exists.

Now we give a proof of Claim 2.

By Lemma 6(i), $x_J \leq x_{J+1}$. Hence $y_J < 4x_1$. By Lemma 8(i), this implies

$$\pi[0, 4x_1] > 1/2. \tag{21}$$

Let $d = D/\beta$. There are six mutually exclusive and exhaustive cases.

Case 1 : $x_1 \leq \frac{6W}{d}\Delta$. Hence $2x_1 \leq int(12W/d)\Delta$. By concavity of w, $w(2x_1, \Delta) \geq w[int(12W/d)\Delta, \Delta]$. By Claim 0, this implies

$$w(2x_1, \Delta) > (R\rho\beta)^{int(12W/d)-1} w(\Delta). \tag{22}$$

Case 2 : $x_1 > \frac{6W}{d}\Delta$ and $w(8x_1) - w(4x_1) \geq d/3$. It follows that

$$w(2x_1, \Delta) > \frac{w(8x_1) - w(4x_1)}{4x_1}\Delta > \frac{d}{12x_1}\Delta > \frac{d}{12Wu'(0)}w(\Delta), \tag{23}$$

where the last inequality follows from Lemma 9(i).

Case 3 : $x_1 > \frac{6W}{d}\Delta$, $w(8x_1) - w(4x_1) < d/3$, and $w(4x_1) \leq 2d/3$. Hence $w(8x_1) < d$. This case is impossible: Lemma 15(ii) and (21) imply $w(8x_1) \geq D/\beta = d$.

The remaining 3 cases involve assumptions either about $x_1/2$ or $(x_1 - \Delta)/2$, whichever is in B_Δ. The arguments are written under the assumption that $x_1/2 \in B_\Delta$. If not, then $(x_1 - \Delta)/2 \in B_\Delta$ would appear in place of $x_1/2$.

Case 4 : $x_1 > \frac{6W}{d}\Delta$, $w(8x_1) - w(4x_1) < d/3$, $w(4x_1) > 2d/3$, $w(4x_1) - w(x_1/2) \leq 2d/3$, and $\pi[x_1/2, 4x_1] \geq 1/4$. Hence $w(8x_1) - w(x_1/2) < d$. This case is impossible: Lemma 15(ii) and $\pi[x_1/2, 4x_1] \geq 1/4$ imply $w(8x_1) - w(x_1/2) \geq D/\beta = d$.

Case 5 : $x_1 > \frac{6W}{d}\Delta$, $w(8x_1) - w(4x_1) < d/3$, $w(4x_1) > 2d/3$, $w(4x_1) - w(x_1/2) \leq 2d/3$, and $\pi[x_1/2, 4x_1] < 1/4$. By (21) and the hypothesis, $\pi[0, x_1/2] > 1/4$. Now we apply Lemma 15(i) with $b_0 = x_1/2$, $b_1 = x_1 + kx_1/2$, and $b_2 = x_1 + (k+1)x_1/2$ for $k = 0, 1$.[4] By $\pi[0, x_1/2] > 1/4$,

$$w[x_1 + (k+1)x_1/2, \Delta] > [R\beta u'(W)/4] w(x_1 + kx_1/2, \Delta).$$

It follows that $w(2x_1, \Delta) > [R\beta u'(W)/4]^2 w(x_1, \Delta)$. This and Lemma 16(i) imply

$$w(2x_1, \Delta) > [R\beta u'(W)/4]^2 [u'(0)]^{-1} w(\Delta). \tag{24}$$

Case 6 : $x_1 > \frac{6W}{d}\Delta$, $w(8x_1) - w(4x_1) < d/3$, $w(4x_1) > 2d/3$, and $w(4x_1) - w(x_1/2) > 2d/3$. Let $a_0 = \min\{x : d/3 > w(x) - w(x_1/2) \geq d/6\}$ and $a_2 =$

[4] If $x_1/2 \notin B_\Delta$, then we can take $b_0 = (x_1 - \Delta)/2$, $b_1 = x_1 + k(x_1 + \Delta)/2$, and $b_2 = x_1 + (k+1)(x_1 + \Delta)/2$ for $k = 0, 1$.

$\max\{x : d/3 > w(4x_1) - w(x) \geq d/6\}$. By the hypothesis and concavity of w, existence of a_0 implies existence of a_2. We claim that a_0 exists. Otherwise, concavity of w implies $w(x_1/2 + \Delta, \Delta) \geq d/3$. But then

$$w(x_1/2) > \frac{x_1}{2}\frac{w(x_1/2+\Delta,\Delta)}{\Delta} \geq \frac{x_1}{2}\frac{d}{3\Delta} > \frac{6W}{d}\frac{d}{6} = W,$$

a contradiction. Note that $4x_1 > a_2 > a_0 > x_1/2$.

First, we have

$$w(a_2,\Delta) > \frac{w(4x_1) - w(a_2)}{4x_1 - a_2}\Delta > \frac{d}{6}\frac{1}{3.5x_1}\Delta > \frac{d}{21Wu'(0)}w(\Delta), \tag{25}$$

where the second inequality follows from the definition of a_2 and the last from Lemma 9(i).

Now let $a_1 = a_2 - (a_0 - x_1/2)$. By definition, $a_2 - a_1 = a_0 - x_1/2$ and $w(a_0) - w(x_1/2) < d/3$. Hence $w(a_2) - w(a_1) < d/3$. Also by definition, $w(4x_1) - w(a_2) < d/3$. It follows that $w(4x_1) - w(a_1) = w(4x_1) - w(a_2) + w(a_2) - w(a_1) < 2d/3$. This and the hypothesis imply $w(8x_1) - w(a_1) < d$. Now either $\pi[a_1, 4x_1] < 1/4$ or $\pi[a_1, 4x_1] \geq 1/4$. If the latter, then by Lemma 15(ii), $w(8x_1) - w(a_1) \geq d$, a contradiction. So $\pi[a_1, 4x_1] < 1/4$. This and (21) imply $\pi[0, a_1] > 1/4$.

By definition, $w(a_0) - w(x_1/2) \geq d/6$. It follows that

$$W > w(x_1/2) > \frac{x_1}{2}\frac{w(a_0) - w(x_1/2)}{a_0 - x_1/2} \geq \frac{dx_1}{12}\frac{1}{a_0 - x_1/2}.$$

This and $a_2 - a_1 = a_0 - x_1/2$ imply $a_2 - a_1 > dx_1/(12W)$. Let

$$i_0 = int[\frac{3x_1/2}{dx_1/(12W)}] = int(18W/d).$$

Then $a_2 > x_1/2$ implies $a_2 + i_0(a_2 - a_1) > x_1/2 + 3x_1/2 = 2x_1$. Now we apply Lemma 15(i) with $b_0 = a_1$, $b_1 = a_2 + k(a_2 - a_1)$, and $b_2 = a_2 + (k+1)(a_2 - a_1)$ for $k = 0, 1, ..., i_0 - 1$. By $\pi[0, a_1] > 1/4$,

$$w[a_2 + (k+1)(a_2 - a_1), \Delta)] > [R\beta u'(W)/4]w[a_2 + k(a_2 - a_1), \Delta].$$

It follows that $w(2x_1, \Delta) > w[a_2 + i_0(a_2 - a_1), \Delta] > [R\beta u'(W)/4]^{i_0} w(a_2, \Delta)$. This and (25) imply

$$w(2x_1,\Delta) > \frac{[R\beta u'(W)/4]^{i_0} d}{21Wu'(0)}w(\Delta). \tag{26}$$

Because cases 3 and 4 are impossible, by (22), (23), (24), and (26), the desired α exists. ■

Proof of Lemma 10

Proof. We first note that $x_1^* \leq x_1$ and $\pi[0, x_1^*) > \rho$ (see the proof of Lemma 6). Now we split the proof into two claims.

Claim 1 *If $x_{n+1} \geq 2x_1^*$, then there exists $c_n > 0$, not dependant on Δ, such that $w_\Delta(y_n, \Delta) > c_n w_\Delta(x_0, \Delta)$.*

We first derive a useful inequality. We apply Lemma 15(i) with $b_0 = x_1^*$, $b_1 = x_{n+1} + kx_n/2$, and $b_2 = x_{n+1} + (k+1)x_n/2$ for $k = 0, 1$. (Note that $x_{n+1} \geq 2x_1^* \Rightarrow x_1^* + x_n/2 \leq x_{n+1}$.)[5] By $\pi[0, x_1^*) > \rho$,

$$w[x_{n+1} + (k+1)x_n/2, \Delta] > R\rho\beta u'(W) w(x_{n+1} + kx_n/2, \Delta).$$

It follows that

$$w(x_{n+1} + x_n, \Delta) > [R\rho\beta u'(W)]^2 w(x_{n+1}, \Delta). \tag{27}$$

Next we discuss three mutually exclusive and exhaustive cases.

Case 1 : $z_n < x_{n+1} + x_n$. So $y_n = x_{n+1} + x_n$. By (27),

$$w(y_n, \Delta) > [R\rho\beta u'(W)]^2 w(x_{n+1}, \Delta). \tag{28}$$

Case 2 : $z_n \geq x_{n+1} + x_n$ and $z_n \leq 3x_n$. Now $y_n = z_n$. We apply Lemma 15(i) with $b_0 = x_1^*$, $b_1 = x_{n+1} + x_n$, and $b_2 = 3x_n$. By $\pi[0, x_1^*) > \rho$, $w(3x_n, \Delta) > R\rho\beta u'(W) w(x_{n+1} + x_n, \Delta)$. This and $z_n \leq 3x_n$ and (27) imply

$$w(y_n, \Delta) > [R\rho\beta u'(W)]^3 w(x_{n+1}, \Delta). \tag{29}$$

Case 3 : $z_n \geq x_{n+1} + x_n$ and $z_n > 3x_n$. Again, $y_n = z_n$. We apply Lemma 15(i) with $b_0 = x_1^*$, $b_1 = z_n - x_n$, and $b_2 = z_n$. (Note that $x_1^* \leq x_1 \Rightarrow x_1^* + 2x_n < z_n$.) By $\pi[0, x_1^*) > \rho$, $w(z_n, \Delta) > R\rho\beta u'(W) w(z_n - x_n, \Delta)$. This and Lemma 16(ii) imply

$$w(y_n, \Delta) > R\rho\beta [u'(W)]^2 w(x_{n+1}, \Delta). \tag{30}$$

By Lemma 16(i), $w(x_{n+1}, \Delta) > [u'(0)]^{-(n+1)} w(x_0, \Delta)$. Hence, by (28), (29), and (30), the desired c_n exists.

[5] If $x_n/2 \notin B_\Delta$, then we can take $b_0 = a$, $b_1 = x_{n+1} + k(x_n + \Delta)/2$, and $b_2 = x_{n+1} + (k+1)(x_n + \Delta)/2$ for $k = 0, 1$. Note that $x_{n+1} \geq 2a$ and $x_n/2 \notin B_\Delta \Rightarrow$ either $x_{n+1} \geq 2a + \Delta$ or $x_{n+1} > x_n \Rightarrow a + (x_n + \Delta)/2 \leq x_{n+1}$.

Claim 2 *If* $x_{n+1} < 2x_1^*$, *then there exists* $C > 0$, *not dependant on* Δ, *such that* $w_\Delta(y_n, \Delta) > Cw_\Delta(\Delta)$.

The proof of this claim is analogous to the proof of Claim 1. We first derive a useful inequality. We apply Lemma 15(i) with $b_0 = x_1^*$, $b_1 = 2x_1^* + kx_n/2$, and $b_2 = 2x_1^* + (k+1)x_n/2$ for $k = 0, 1$. (Note that $x_{n+1} < 2x_1^* \Rightarrow x_n/2 < x_1^*$.)[6] By $\pi[0, x_1^*) > \rho$,

$$w[2x_1^* + (k+1)x_n/2, \Delta] > [R\rho\beta u'(W)]w(2x_1^* + kx_n/2, \Delta).$$

It follows that

$$w(2x_1^* + x_n, \Delta) > [R\rho\beta u'(W)]^2 w(2x_1^*, \Delta). \tag{31}$$

Next we discuss three mutually exclusive and exhaustive cases.

Case 1 : $z_n < x_{n+1} + x_n$. So $y_n = x_{n+1} + x_n$. By $x_{n+1} < 2x_1^*$ and (31),

$$w(y_n, \Delta) > [R\rho\beta u'(W)]^2 w(2x_1^*, \Delta). \tag{32}$$

Case 2 : $z_n \geq x_{n+1} + x_n$ and $z_n \leq 3x_n$. Now $y_n = z_n$. We apply Lemma 15(i) with $b_0 = x_1^*$, $b_1 = 2x_1^* + x_n$, and $b_2 = 3x_n$. By $\pi[0, x_1^*) > \rho$, $w(3x_n, \Delta) > R\rho\beta u'(W)w(2x_1^* + x_n, \Delta)$. This and $z_n \leq 3x_n$ and (31) imply

$$w(y_n, \Delta) > [R\rho\beta u'(W)]^3 w(2x_1^*, \Delta). \tag{33}$$

Case 3 : $z_n \geq x_{n+1} + x_n$ and $z_n > 3x_n$. Again, $y_n = z_n$. We apply Lemma 15(i) with $b_0 = x_1^*$, $b_1 = z_n - x_n$, and $b_2 = z_n$. (Note that $x_1^* \leq x_1 \Rightarrow x_1^* + 2x_n < z_n$.) By $\pi[0, x_1^*) > \rho$, $w(z_n, \Delta) > R\rho\beta u'(W)w(z_n - x_n, \Delta)$. This and Lemma 16(ii) and $x_{n+1} < 2x_1^*$ imply

$$w(y_n, \Delta) > R\rho\beta[u'(W)]^2 w(2x_1^*, \Delta). \tag{34}$$

Note that hypothesis and Lemma 17 imply $w(2x_1^*, \Delta) > C_0 w(\Delta)$. Hence by (32), (33), and (34), the desired C exists. ■

Proof of Lemma 12

Proof. The proof of parts (i)-(iv) is essentially the same as the proof of the corresponding parts of Lemma 5. Now consider part (v). If $x - \tilde{p}(x, m) \in Q$, then by continuity of $\tilde{p}$ (see the proof of Lemma 1), there exist open

[6] If $x_n/2 \notin B_\Delta$, then we can take $b_0 = a$, $b_1 = 2a + k(x_n + \Delta)/2$, and $b_2 = 2a + (k+1)(x_n + \Delta)/2$ for $k = 0, 1$. Note that $x_{n+1} < 2a \Rightarrow (x_n + \Delta)/2 \leq a$.

sets $Q_1 \ni x$ and $Q_2 \ni m$ in $[0, B]$ such that $(x', m') \in Q_1 \times Q_2$ implies $x' - \tilde{p}(x', m') \in Q$. By the definition of *supp* μ, $x, m \in$ *supp* μ implies $\mu Q_1, \mu Q_2 > 0$. It follows that $\mu Q > 0$. The other part of part (v) follows in the same way. ■

Proof of Lemma 13

Proof. Assume by contradiction that $\min\{x : \mu[0, x] = 1\} = a < B$. Note that $a \in$ *supp* μ. Either $\tilde{p}(a, a) = 0$ or $\tilde{p}(a, a) > 0$. If the latter, then $\tilde{p}(a, a) + a > a$. Because $a \in$ *supp* μ, by Lemma 12(v), this implies $\mu(a, B] > 0$, a contradiction. So $\tilde{p}(a, a) = 0$. This implies $v'_-(a) \geq u'(0)v'_+(a)$, where $v'_-(a)$ and $v'_+(a)$ denote the left and right derivative of v at a respectively. (Note that if $u'(0) = \infty$, then a contradiction follows immediately.) Now either $\mu\{a\} < 1/4$ or $\mu\{a\} \geq 1/4$. Assume the latter. Because $\tilde{p}(a, a) = 0$, $\tilde{f}(a + \varepsilon, a) - \tilde{f}(a, a) \geq u[\beta v(a + \varepsilon, \varepsilon)]$ for $\varepsilon > 0$. Then (v, μ) being a steady state implies $v(a+\varepsilon, \varepsilon) > R\mu(a)u[\beta v(a+\varepsilon, \varepsilon)] \geq (R/4)u[\beta v(a+\varepsilon, \varepsilon)]$. Because $(R\beta/4)u'(0) > 1$, the equation $x = (R/4)u(\beta x)$ has a unique positive solution for x. Therefore, $v(a + \varepsilon, \varepsilon) > (R/4)u[\beta v(a + \varepsilon, \varepsilon)]$ implies that $v(a + \varepsilon, \varepsilon)$ is bounded below by that positive solution as $\varepsilon \to 0$. But then v is discontinuous at a, a contradiction. So $\mu\{a\} < 1/4$ and there exists $z < a$ with $\mu[0, z] \geq 3/4$. Let $p = \min\{p(a, m) : m \in [0, z]\}$. By Lemma 12(iv) and continuity of $p(.,.)$, $p > 0$. Fix $\varepsilon \in (0, p)$ and let $m \in [0, z]$. We have $\tilde{f}(a + \varepsilon, m) - \tilde{f}(a, m) \geq \beta v(a - p + \varepsilon, \varepsilon) > \beta v'_-(a)\varepsilon$. It follows that $v(a + \varepsilon, \varepsilon) > R\mu[0, z]\beta v'_-(a)\varepsilon \geq (R\beta 3/4)u'(0)v'_+(a)\varepsilon > v'_+(a)\varepsilon$. (Here the second inequality follows from $v'_-(a) \geq u'(0)v'_+(a)$.) But this contradicts strict concavity of v. ■

References

[1] C. Aliprantis, K. Border, Infinite Dimensional Analysis, a Hitchhiker's Guide, Second Edition, Springer-Verlag, New York, 1999.

[2] A. Billingsley, Convergence of Probability Measures, John Wiley & Sons, New York, 1968.

[3] P. Huber, Robust Statistics, John Wiley & Sons, New York, 1981.

[4] N. Kiyotaki, R. Wright, On money as a medium of exchange, J. Polit. Economy 97 (1989), 927-54.

[5] E. McShane, T. Botts, Real Analysis, D. Van Nostrand Company, Princeton, 1959.

[6] M. Molico, The distribution and prices in searching equilibrium, Ph.D. Thesis, University of Pennsylvania, Philadelphia, 1997.

[7] S. Shi, Money and prices: a model of search and bargaining, J. Econom Theory 67 (1995), 467-98.

[8] A. Taber, N. Wallace, A matching model with bounded holdings of indivisible money, Internal. Econom. Rev. 40 (1999), 961-84.

[9] A. Trejos, R. Wright, Search, bargaining, money and prices, J. Polit. Economy 103 (1995), 118-41.

[10] N. Wallace, T. Zhu, A commodity-money refinement in matching models, J. Econom Theory, forthcoming.

[11] T. Zhu, Existence of monetary steady states in a matching model: indivisible money, J. Econom Theory 112 (2003), 307–324.