Pairwise Monitoring, Information Revelation, and Message Trading

Tao Zhu*and Eliot Maenner†

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Abstract

In a random-matching risk-sharing model, risk-sharing actions are only monitored by the pair in the meeting, while each agent can send his message about actions to the public. A risk-sharing outcome and the message about the outcome are determined simultaneously, allowing the message and outcome to be traded with each other. We characterize conditions in terms of agents' continuation values for a certain outcome to occur. We demonstrate when there is a folk theorem and when there is not. We show that public messages, as an information revelation technology, treat money as a special case.

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1 Introduction

The informational roles of a variety of real-life objects motivate a strand of literature to formulate and investigate technologies that reveal information about past actions in random matching models. Some of the well known technologies are labels (Okuno-Fujiwara and Postlewaite [27] and Kandori [16]), money (e.g., Kiyotaki and Wright [18, 19]), and memory (Kocherlakota [20]): labels are broadly interpreted as reputation, social status, credit cards, etc;

*Hong Kong University of Science and Technology, <taozhu@ust.hk>

†Federal Communications Commission, <Eliot.Maenner@fcc.gov>

money is interpreted as currency; and memory is an idealized technology designed to be superior to money.[1] With pairwise monitoring—actions taken in each pairwise meeting are observed only by the pair of agents in the meeting, Kandori [16] and Ellison [9] establish folk theorems with contagious equilibria for a class of prisoners' dilemma stage games. But when the size of the population is increased, the discount factor must increase in order to maintain cooperation, making cooperation harder in a larger population and impossible in an infinite limit. This well-known result demonstrates the potential usefulness of information revelation technologies to sustain cooperation by overcoming the informational friction of pairwise monitoring among a large population and, in turn, provides a foundation for understanding the roles of related real-life objects in environments where many people are indirectly linked and there is limited observation of past actions.

In this paper, we explore two interrelated questions for information revelation technologies that do not eliminate pairwise monitoring. First, what is the extent of cooperation that can be sustained by these technologies? Secondly, can one technology be identified as a special case of another whose information-revelation function is well understood? Because pairwise monitoring remains as a primitive, the technologies in consideration can only help a third party to infer actions in a meeting. Such technologies include money (representing currency), but neither labels nor memory,[2] and would be well-suited, if appropriately formulated, to represent real-life objects (other than currency) that have analogous properties. Notably, checks and credit cards, commonly substituted for currency, seem to be real-life objects of this sort: if paying by currency does not alter that only a buyer and seller observe the transfer of a good between them, then paying by a check or a card seems not

[1] In monetary economics, the informational role of money is first exploited by Ostroy [29], describing money as a record keeping device, a term widely used in the subsequent literature even when trade is not pairwise (e.g., Townsend [34] and Kocherlakota [20]); the first infinite-horizon model with pairwise monetary trade is formulated by Diamond [8]. In [20] memory is also defined when matching is not pairwise. Kocherlakota and Wallace [22] formulate a stochastic version of memory; similar technologies also appear in other applications of matching models (e.g., Dixit [7], and Tirole [33]). Matsui [26] compares matching models for labels, money, and memory.

[2] With labels, the information about actions in each pairwise meeting is truthfully transmitted to (or, equivalently, these actions are observed by) some institution that updates the labels based on the information; two agents in the meeting can observe only each other's label. With memory, the information about one's past actions is accessible to (or, equivalently, these actions are observed by) his direct and indirect matching partners.

to either. Determining the extent of cooperation (the first question) helps to demarcate constraints on cooperation caused by pairwise monitoring from constraints caused by technology-specific physical or institutional characteristics; after all, money and a technology representing credit cards ought to have different characteristics. It also opens the door to future explorations; for example, if money is inefficient in that it sustains less cooperation than alternative technologies, what changes to the model would make money efficient? Identification of one technology with another (the second question) helps to define an exact sense in which the former (e.g., money) reveals information about past actions.

For our purposes, we study an idealized technology—public messages, i.e., each agent can send messages about actions in his present meeting to the public. Public messages are not new in repeated games with private monitoring when the stage game is played by a fixed set of agents. Indeed, Ben-Porath and Kahneman [3] establish a folk theorem for a class of such games with partial (i.e., private but perfect) monitoring.[3] As the distinct feature in our employment of public messages, people are able to exchange messages with other transferable objects of value—message trading. In other words, we consider not only the individual incentive constraint for certain messages to be transmitted but also the multilateral, bilateral in our context, constraint.

To see how message trading may arise and restrict the effectiveness of messages in sustaining cooperation, consider a one-shot cake-sharing example with two players, A and B, and an arbitrator. The arbitrator gives one unit of the cake to player A and announces a reward scheme designed to yield an even split based on each player's message about the shares of the cake they respectively receive. Then the players share the cake and create messages with some device (e.g., written on paper) in a spot without the arbitrator. After receiving each player's message, the arbitrator allocates rewards. Think of a scheme that gives B a constant reward. Absent of message trading, B does not have an incentive to lie. With B telling the truth, if the reward to A suitably varies with B's message, then A will transfer half the cake to B. But message trading undermines this scheme: What if A, deviating from transferring half, eats 2/3 of the cake, and then offers the remaining 1/3 to

[3] There are also folk theorems for a class of such games with imperfect private monitoring (e.g., Compte [6] and Kandori and Matsushima [17]). Mailath and Samuelson [24] provide a general reference for repeated games.

B in exchange for a message that gives A the highest reward, a mutually beneficial exchange?

This example also relates messages and message trading to money and monetary exchange. Suppose the arbitrator gives each player some tokens (money) before they enter the room and allocates awards according to players' post-sharing token holdings instead of messages. Tokens have the same informational role as messages. So we treat message trading as a generalization of monetary exchange in matching models (e.g., Kocherlakota [20], Shi [32], and Trejos and Wright [35]); that is, message trading is governed by a game form (e.g., the ultimatum game) and, if by playing the game form players reach an agreement to exchange some cake for both players' messages, then they are committed to carrying it out on the spot.[4]

In the example, having the creation of messages be part of the cake-sharing game helps to relate money to messages, but is not necessary for message trading. Suppose the cake sharing and creation of messages occur sequentially in two distinct spots. Message trading may arise when the arbitrator is not at either spot and some valuable object is available at the message-creation spot.[5] This sequential setup could represent some real-life uses of messages (e.g., customer reviews and credit scores). In the conclusion, we discuss how our analysis can be largely extended to this setup and the emergence of new conceptual issues (e.g., hard evidence).

Formally, we study a (repeated) random-matching risk-sharing model with a continuum of agents. One agent in each pair is randomly chosen to receive some cake; who is chosen is public to all agents. The size of the cake can be a random variable whose value is either one or zero. The realization of the cake size (when the size is random) and how the agents divide it are pairwise monitored and are described in each agent's public message. We adapt perfect public equilibrium in a natural way. To capture the essence of message trading—two parties seeking mutually beneficial outcomes—we introduce two refinements, coalition proofness (CP) and renegotiation proofness (RP). Each refinement is static in that it is only applied to plays in the trading game of each meeting.

[4]By way of making message trading possible, this on-the-spot commitment restricts the effectiveness of messages in sustaining cooperation; it does not extend to actions outside the spot.

[5]If the arbitrator is at the message-creation spot, then he can deter message trading, resembling the setup in Ben-Porath and Kahneman [3] with public incrimination in a collective meeting.

We characterize the magnitudes of rewards and punishments, in terms of agents' continuation values, necessary and sufficient for a certain transfer to occur in the meeting. With CP, it is necessary to reward an agent for him to reveal that his meeting partner makes a transfer smaller than the one specified by the equilibrium. With RP, when the size of the cake is random, the model has a property akin to the familiar incentive compatibility in the centralized risk-sharing models with private information (e.g., Atkeson and Lucas [2] and Green [11]).[6]

With CP there is a folk theorem. With RP, if the size of the cake is not random (always equal to one), then the folk theorem holds but requires a higher greatest lower bound on the discount factor than the CP counterpart; otherwise, the folk theorem does not hold and the welfare loss does not vanish as the discount factor approaches one. But some continuity is preserved from randomness to certainty.

Money is a special case of public messages that supports a more restrictive message space and, hence, no more equilibrium allocations than public messages. Following Kocherlakota [20], it is well understood that when there is memory, money is not essential. When pairwise monitoring is a primitive, our result suggests a complementary view: If public messages are costless to maintain, then money is not essential; otherwise, money is essential when it is not dominated by another information revelation technology in terms of social benefits net costs.

This view of essentiality seems useful to model non-cash payment methods such as checks and credit cards, which, as elaborated on below, involve two tiers of information revelation. The primary tier is for transfers of goods; the secondary tier, a tier necessary for the credit feature of these payment methods, is for related payment histories. To see relevance of this point, one may think of mapping goods paid by credit cards into cash-in-advance models as credit goods or cash goods, or modelling a cashless economy in which goods are all paid by non-cash payment methods.[7]

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[6] In our model there is no private information (the endowment realization is pairwise monitored), and the redistribution of resources is decentralized (pairwise) and not public (pairwise monitored).

[7] Woodford [38] provides a well-known model for a cashless economy, i.e., a cash-in-advance model with 100% credit goods, where no payment method is needed. Notably, credit goods are interpreted differently by users of cash-in-advance models. For example, they may be non-market goods (cf. Lucas and Stokey [23]); they may be goods paid by checks (cf. Aiyagari and Eckstein [1]); they may exclude goods paid by checks (cf. Hodrick,

2 The basic model

Time is discrete, dated $t \geq 1$. There is a nonatomic measure set I of infinitely lived agents. At the start of each date, each agent is subject to an i.i.d. shock that determines his type: with equal probability he becomes a buyer or a seller. This *type realization* is *public information*—information known to all agents. Then each buyer is randomly matched with a seller. The *matching realization*, i.e., the identity of each agent in each meeting, is also public information.

In each meeting, the buyer is endowed with 0 units of a good; the seller's endowment of the good is an i.i.d. variable: it is 1 with probability $\rho \in (0, 1]$ and 0 with probability $1-\rho$. When $\rho < 1$ the *seller's endowment realization* is *pairwise public information*—meeting-specific information observed only by the pair of agents in the meeting (i.e., the realization is pairwise monitored). After the endowment realization, the meeting consists of two consecutive stages.

Stage 1. The seller chooses to consume any part of his endowment. This consumption is pairwise public information.

Stage 2. The buyer and seller play an extensive game form prescribed by a given trading mechanism as defined below. In the game form, there is no move of nature; each history is pairwise public information; and each terminal node is an outcome. An *outcome*, denoted (y, r), consists of a transfer of y units of the good (from the seller to buyer) and a report r. A *report*, denoted $r = (r_b^1, r_b^2, r_s^1, r_s^2) \in \{0, 1\} \times [0, 1] \times \{0, 1\} \times [0, 1]$, consists of messages that the buyer and seller input into a *reporting device* available in the meeting, where r_b^1 and r_b^2 (r_s^1 and r_s^2, respectively) are the buyer's message (the seller's, respectively) regarding the seller's endowment realization and the transfer of the good, respectively. In any non-terminal play of the game form, the good is not transferred or consumed and there is no input into the reporting device. There is a kind of non terminal play called *autarky* which means both agents simultaneously announce an element in $\{0, 1\} \times [0, 1]$. Autarky is followed by terminal nodes each of which is referred to as an *autarky outcome*, i.e., $y = 0$ and r with the elements that agents announced in autarky. A terminal node that does not follow autarky is referred to as a *trading outcome*.

After the meeting, the report becomes public information (no other pair-

Kocherlakota, and Lucas [14], who map M2 to the stock of money in their model); and they seem to include market goods in [38].

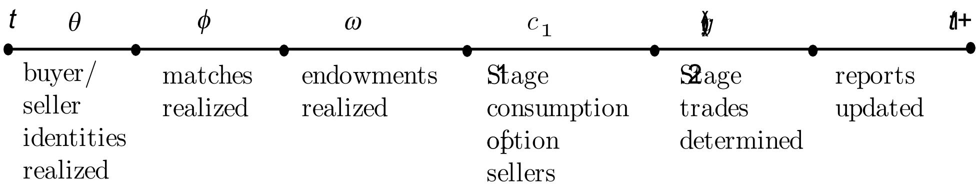

Figure 1: Sequence of events.

wise public information of the meeting becomes public information); the good is consumed; any unconsumed good perishes; and then the date is over.

From consuming c during the date, the seller's period utility is c (his stage-1 consumption is part of c), and the buyer's is $u(c)$ with $u(0) = 0$, $u' > 0$, and $u'' < 0$.[8] To make a positive transfer ex-ante more desirable than a zero transfer, let

$$q^* \equiv \arg\max[u(q) + (1 - q)] > 0. \tag{1}$$

Each agent maximizes the expected discounted utility normalized by $1 - \delta$, where $\delta \in (0, 1)$ is the discount factor.

The following notation is used throughout. Let $\theta_{i,t} = 0$ if agent $i \in I$ is a buyer at t and $\theta_{i,t} = 1$ if a seller. Let $\phi_{i,t} \in I$ be i's meeting partner at t. For the date-t meeting between i and $\phi_{i,t}$, let $\omega_{i,t} \in \{0, 1\}$ be the seller's endowment realization, let $c_{1,i,t} \in [0, \omega_{i,t}]$ be the seller's stage-1 consumption, and let $r_{i,t} = (r^1_{i,b,t}, r^2_{i,b,t}, r^1_{i,s,t}, r^2_{i,s,t})$ be the report of the meeting. (Notice that $\omega_{k,t} = \omega_{j,t}$, $c_{1,k,t} = c_{1,j,t}$, and $r_{k,t} = r_{j,t}$ if $j = \phi_{k,t}$.) For $x \in \{\theta, \phi, \omega, r\}$, let $x^\tau_i = (x_{i,0}, ..., x_{i,\tau})$ with $x_{i,0} = i$ for $\tau \geq 0$.

Definition 1 *A trading mechanism T is a collection of mappings $(T_{i,t})_{i \in I, t \geq 1}$ such that for any $(\theta^t_i, \phi^t_i, r^{t-1}_i)_{i \in I}$, k, $\omega_{k,t}$, and $c_{1,k,t}$, the mapping $T_{k,t}$ prescribes to agents k and $\phi_{k,t}$ an extensive game form at stage 2 in their date-t meeting.*

[8] Linearity of the seller's utility is without loss of generality. When the buyer and seller have concave and strictly increasing utility functions, say, u_b and u_s, respectively, we can make the transformation by setting $u(c) = u_b(\nu^{-1}(c))$ with $\nu(c) = u_s(1) - u_s(1 - c)$. Also, we can extend our analysis to weakly concave u. Strict concavity simplifies exposition of Proposition 4 and its proof, but it is not substantial for this proposition and any subsequent result.

By definition, $T_{k,t} = T_{j,t}$, $j = \phi_{k,t}$. (As convention, i stands for a generic agent on the set I; k serves the same role when i is employed in the same context.) This definition follows closely the one in Kocherlakota [20]. Following [20], the prescribed game form satisfies *no commitment* in that following any $(\omega_{k,t}, c_{1,k,t})$, k has a sequence of actions leading to autarky, independent of any sequence of actions chosen by $\phi_{k,t}$ (e.g., ultimatum games); also, the game form ends before the date is over.[9]

The matching process, the endowment process, the seller's action at stage 1, the trading mechanism T applied to stage 2, and the preferences define a game. We restrict our attention to what we call quasi-public strategies, adapted from public strategies in a natural way (consistent with the ones adopted in [20]).

Definition 2 *A strategy $\sigma_k = (\sigma_{k,t})_{t \geq 1}$ of agent k is a quasi-public strategy if for any t, when k moves at any information set at his date-t meeting, $\sigma_{k,t}$ only conditions on the public information, i.e., $(\theta_i^t, \phi_i^t, r_i^{t-1})_{i \in I}$, and the meeting-specific pairwise public information, i.e., $\omega_{k,t}$ and actions already taken by k and $\phi_{k,t}$ during the meeting.*

If σ_k is a quasi-public strategy, then $\sigma_{k,t}$ specifies the same action on two information sets of k in his date-t meeting which differ only in the pairwise public information pertaining to his meetings before t.[10]

Our equilibrium concept is perfect equilibrium and we only consider pure strategy equilibrium (also consistent with [20]).[11]

Definition 3 *Given a trading mechanism T, a profile of strategies $\sigma = (\sigma_i)_{i \in I}$ is an equilibrium if for any i the strategy σ_i is a pure quasi-public strategy and if evaluated at any history of the game σ determines a Nash equilibrium.*

[9] This can be achieved if the game form is a multistage form with a finite number of stages or, even with an infinite number of stages if the duration of a stage shrinks at a geometric rate.

[10] It would be natural to let $\sigma_{k,t}$ also depend on pairwise public information of meetings between k and $\phi_{k,t}$ before t. As I is a continuum, adding such dependence does not change any result.

[11] Pure strategy certainly is not restrictive for the folk-theorem results. With a pure strategy, no move of nature in the trading game form, and deterministic outcomes of the trading game form, we avoid some technical issues seemingly not critical to other results. Refinements in Definitions 5 and 6 do require an equilibrium outcome not to be dominated by lotteries over outcomes.

Definition 4 *An allocation f is a collection of mappings $(f_{i,t})_{i\in I,t\geq 1}$ such that for any $(\theta_i^t, \phi_i^t, \omega_i^{t-1})_{i\in I}$, k, and $\omega_{k,t}$, the mapping $f_{k,t}$ prescribes to agents k and $\phi_{k,t}$ a transfer of goods (i.e., $f_{k,t}((\theta_i^t, \phi_i^t, \omega_i^{t-1})_{i\in I}, \omega_{k,t})$) in their date-$t$ meeting.*

By definition, $f_{k,t} = f_{j,t}$, $j = \phi_{k,t}$. Also by definition, the transfer prescribed by $f_{k,t}$ depends only on the history of the physical environment (in particular, not on how the transfer is monitored or any technology that facilitates monitoring).

An allocation f is supported by an equilibrium σ if for any $(\theta_i^t, \phi_i^t, \omega_i^{t-1})_{i\in I}$, k, and $\omega_{k,t}$, the transfer specified by $f_{k,t}$ coincides with the transfer of goods determined by on-path plays of σ.

3 Perfect monitoring benchmark

To examine the role of public messages to sustain cooperation under pairwise monitoring, we introduce the following variant of the basic model as the benchmark. In each pairwise meeting, stage 1 is unchanged; at stage 2 the seller chooses a transfer and there are no reports; and the seller's endowment realization and the transfer of the good become public information after the meeting. We refer to this no-report perfect-monitoring model as the *perfect-monitoring benchmark.* The equilibrium concept is adapted from Definition 3. Specifically, T is redundant and when $\theta_{k,t} = 1$, $\sigma_{k,t}$ conditions on $(\theta_i^t, \phi_i^t, \omega_i^{t-1}, q_i^{t-1})_{i\in I}$, $\omega_{k,t}$, and k's own action already taken in the meeting, where $q_i^{t-1} = (q_{i,0}, ..., q_{i,t-1})$, $q_{i,\tau}$ is the transfer of the good in the date-τ meeting between i and $\phi_{i,\tau}$, and $q_{i,0} = i$.

Proposition 1 *An allocation f is supported by an equilibrium in the basic model if and only if it is supported by an equilibrium in the perfect-monitoring benchmark.*

Proof. For the "if" part, let f be supported by an equilibrium σ in the benchmark. To describe T' and σ' in the basic model, fix $\gamma_t \equiv (\theta_i^t, \phi_i^t, r_i^{t-1})_{i\in I}$, k, and $\omega_{k,t}$. Given γ_t, set $\omega_{i,\tau} = r_{i,b,\tau}^1$ if $1 \leq \tau < t$ and let $\mu_t = (\theta_i^t, \phi_i^t, \omega_i^{t-1})_{i\in I}$; also let $\Lambda_t = 0$ if $r_{i,b,\tau}^2 \cdot r_{i,s,\tau}^2 = 0$ for some i and $1 \leq \tau < t$, and let $\Lambda_t = 1$ otherwise. In the game form prescribed by $T'_{k,t}$ following $c_{1,k,t}$, k and $j = \phi_{k,t}$ simultaneously say Yes or No. If both say Yes and $1 - c_{1,k,t} \leq y_{k,t} \equiv f_{k,t}(\mu_t, \omega_{k,t})$, then the outcome is $(y_{k,t}, r)$ with $r = (\omega_{k,t}, 1, \omega_{k,t}, 1)$; otherwise autarky is reached. Actions specified by $\sigma'_{k,t}$ are as follows. At stage 1 choose $c_{1,k,t} = 0$ if $\theta_{k,t} = 1$. At stage 2 say Yes iff $\Lambda_t = 1$; in autarky announce $(\omega_{k,t}, 0)$. (We only describe $\sigma'_{k,t}$ because of k's representative role.) To see that σ' is an equilibrium, the case worth of checking is that given $(\gamma_t, \omega_{k,t}, c_{1,k,t})$ with $\theta_{k,t} = \Lambda_t = 1$ and $y_{k,t} \geq 1 - c_{1,k,t} > 0$, k does not gain by saying No if all other agents do not deviate from t on and there is no further deviation of k. Saying Yes in σ' gives k the same payoff as his on-path play in σ given $(\mu_t, \omega_{k,t})$ (i.e., the payoff implied by f given $(\mu_t, \omega_{k,t})$). Saying No in σ' gives k the payoff $(1-\delta)\cdot 1 + \delta \cdot 0.5\rho$ (k's No triggers the global autarky from t+1—each agent eats his own endowment forever—so k's start-of-t+1 continuation value is 0.5ρ); deviating to $q_{k,t} = 0$ in σ gives k the payoff $(1-\delta)\cdot 1 + \delta \cdot v$ for some $v \geq 0.5\rho$ (k can guarantee himself 0.5ρ). Because σ is an equilibrium, k does not gain by saying No in σ'. The proof of the "only-if" part is in the appendix. ■

The "only-if" part of Proposition 1 says that perfect monitoring sets an upper bound on what public messages can achieve.[12] The logic for this part is simple. For an agent, on-path payoffs in σ' in the benchmark and in σ in the basic model are the same (i.e., the payoffs implied by f). With perfect monitoring, his continuation value following a deviation in σ' can be ensured not to exceed his continuation value following a deviation in σ, while the deviation in σ' need not trigger the global autarky (which is not the case for σ' in the proof of the "if" part).

For σ' in the proof of the "if" part, when there is no defector up to the start of t but $c_{1,k,t} \in (1 - y_{k,t}, 1]$, $T_{k,t}$ only admits the autarky outcomes even though for k and j these outcomes are Pareto dominated by $y \in [0, 1 - c_{1,k,t}]$ and $r = (1,1,1,1)$.[13] This illustrates weakness of Definition-3 equilibrium in capturing the essence of message trading—two parties seek mutually beneficial outcomes—in the absence of further restrictions on T, motivating refinements in the next section.

[12] In spirit, the "only-if" part resembles the money-is-memory result in Kocherlakota [20], which generalizes the result in Townsend [20] that compares money with a technology called communication, serving the function of perfect monitoring. Further comments follow Proposition 10.

[13] To establish the converse of the money-is-memory result, Kocherlakota [21] uses a similar trick. In particular, while a seller with on-path money holdings and a buyer with off-path holdings have incentives to trade (the seller can keep post-meeting holdings on path and the buyer can get goods), the game form only admits autarky outcomes.

4 Refinements and partial characterization

Here we introduce two refinements for a Definition-3 equilibrium σ, coalition proofness (CP) and renegotiation proofness (RP). Each refinement is static in that it only places restrictions on the strategies of agents k and $j = \phi_{k,t}$ played in the game form prescribed by $T_{k,t}$ (for any k and t). For each refinement, we also characterize conditions on the continuation values for a transfer to occur between k and j.

We begin with CP, which is the Coase Theorem in our context. That is, $\sigma_{k,t}$ and $\sigma_{j,t}$ lead k and j to a pairwise efficient outcome in the game form.[14] An outcome of the game form is *pairwise efficient* if it is not pairwise Pareto dominated (in terms of payoffs implied by σ) by any *lottery* over the set of all (physically) feasible outcomes, regardless of being admitted by the game form or not.

Definition 5 *Given a trading mechanism T, an equilibrium σ (Definition 3) is a CP equilibrium if for any $(\theta_i^t, \phi_i^t, r_i^{t-1})_{i \in I}$, k, $\omega_{k,t}$, and $c_{1,k,t}$, $\sigma_{k,t}$ and $\sigma_{j,t}$ ($j = \phi_{k,t}$) specify a pairwise efficient outcome in the game form prescribed by $T_{k,t}$.*

The next result says that in order to induce the seller to make a transfer in a CP equilibrium, it is necessary not just to punish the seller if his deviation to a smaller transfer is revealed (i.e., the conventional participation constraint of the seller) but also to reward the buyer to reveal that deviation.[15]

Proposition 2 *Let σ be a CP equilibrium. Fix t, $(\theta_i^t, \phi_i^t, r_i^{t-1})_{i \in I}$, and k with $\theta_{k,t} = 0$ (k is a buyer). Suppose that $\sigma_{k,t}$ and $\sigma_{j,t}$ ($j = \phi_{k,t}$) specify an outcome (y^*, r^*) with $y^* > 0$ when $\omega_{k,t} = 1$, and an outcome (y°, r°) following some $c_{1,k,t} = c > 1 - y^*$. Let $v_i(r)$ be agent i's start of $t+1$ continuation value implied by a report r from the meeting, $i \in \{k, j\}$. Then $v_k(r^\circ) - v_k(r^*) > 0$ and $v_j(r^*) - v_j(r^\circ) \geq \alpha(\delta)(y^* - y^\circ)$, where $\alpha(\delta) = \delta^{-1}(1 - \delta)$.*

Proof. If $v_j(r^*) - v_j(r^\circ) < \alpha(\delta)(y^* - y^\circ)$ then j can gain by deviating to consume c at stage 1 and reach (y°, r°) at stage 2. If $v_k(r^\circ) - v_k(r^*) \leq 0$

[14] Such a solution concept is used in the literature on labor and monetary matching and search models; e.g., Hall [13], Hu, Kennan and Wallace [15], and Zhu and Wallace [39].

[15] So the equilibrium in the proof of the "if" part of Proposition 1 cannot be a CP equilibrium.

then given $v_j(r^*) - v_j(r^\circ) > 0$ (by $c > 1 - y^*$, $y^\circ < y^*$), (y°, r°) is Pareto dominated by (y°, r^*). ■

It is helpful to cast Proposition 2 in the context of a one-shot game followed by payoffs measured in utils (i.e. the continuation values) that are allocated by a third party (e.g., the cake-sharing game with the arbitrator in the introduction). Let the buyer and seller have a discount factor $1/\alpha(\delta)$ when the third party allocates payoffs and normalize both agents' on-path payoffs to zero. For y^* to be transferred, $v_k(r^\circ) - v_k(r^*) > 0$ gives a lower bound on the third party's payoff capacity to reward the buyer, i.e., any small but positive number; $v_j(r^*) - v_j(r^\circ) \geq \alpha(\delta)y^*$ (set $y^\circ = 0$) gives a lower bound on the third party's payoff capacity to punish the seller, i.e., $\alpha(\delta)y^*$. As it turns out, the third party's capacities reaching these bounds is sufficient for y^* to be transferred. Here we do not prove sufficiency—it is a natural ingredient in the folk theorem with CP equilibrium in the next section.

Next we turn to RP, which restricts payoffs of pairwise Pareto dominated outcomes in the game form prescribed by $T_{k,t}$. The idea is that if one agent, say j, deviates (in a CP equilibrium) and a pairwise dominated outcome is reached, k and j will renegotiate to an efficient term. Furthermore, if the renegotiated term makes j better off than the one implied by $\sigma_{k,t}$ and $\sigma_{j,t}$, j would deliberately deviate in the first instance. This is the same idea that motivates RP in models in the Nash-implementation literature. To determine how renegotiation selects an efficient term, we follow a standard treatment in that literature (cf. Maskin and Moore [25]) by letting renegotiation be a mapping that operates on a class of sets. A set in the class consists of lotteries over all feasible outcomes at the point from which an inefficient outcome of the prescribed game form is reached. The mapping selects from the set a lottery, possibly degenerate, that is pairwise efficient (i.e., not pairwise Pareto dominated by another lottery in the set). Following a widespread convention, we restrict attention to a special mapping, one that selects a lottery to maximize the Nash product with equal weights on the buyer and seller, and with the reached inefficient outcome serving as the disagreement point.[16]

[16] Nash bargaining is widely adopted in the literature on labor and monetary matching and search models. Here, if we explicitly formulate the renegotiation process as the familiar alternating-offer bargaining game with an exogenous break-down probability, then, as is well known, the Nash solution is the limit of the equilibrium outcomes of the bargaining

Definition 6 *Given a trading mechanism T, a CP equilibrium σ is a RP equilibrium if for any $(\theta_i^t, \phi_i^t, r_i^{t-1})_{i\in I}$, k, $\omega_{k,t}$, and $c_{1,k,t}$, $\sigma_{k,t}$ and $\sigma_{j,t}$ ($j = \phi_{k,t}$) remain an equilibrium in the game form prescribed by $T_{k,t}$ when each pairwise inefficient outcome in terminal nodes is replaced with a lottery over all feasible outcomes that maximizes the Nash product of k and j's payoffs with the inefficient outcome serving as the disagreement point.*

Renegotiation-proofness implies a property that resembles the familiar incentive compatibility. That is, if the endowment realization is stochastic (i.e., $\rho < 1$), then to prevent the seller from deviating when he is endowed, he will be punished and the buyer will be rewarded when he turns out to be not endowed. In the context of the above one-shot game, this means that the third party's payoff capacities to reward the buyer and punish the seller must be utilized when the seller is not endowed. Formally we have the following.

Proposition 3 *Let $\rho < 1$. Let σ be a RP equilibrium. Fix t, $(\theta_i^t, \phi_i^t, r_i^{t-1})_{i\in I}$, and k with $\theta_{k,t} = 0$. Let j, (y^*, r^*), and $v_i(.)$ be given by Proposition 2. Suppose that $\sigma_{k,t}$ and $\sigma_{j,t}$ specify an outcome $(0, r)$ when $\omega_{k,t} = 0$. Then $v_k(r) - v_k(r^*) > 0$ and $v_j(r^*) - v_j(r) \geq \alpha(\delta)y^*$.*

Proof. Let (y°, r°) be the one in Proposition 2 when $\omega_{k,t} = c_{1,k,t} = 1$ (so $y^\circ = 0$). By Proposition 2, it suffices to show that $\Delta_k \equiv v_k(r^\circ) - v_k(r) = 0$ and $\Delta_j \equiv v_j(r) - v_j(r^\circ) = 0$. Suppose the converse so $\Delta_k\Delta_j > 0$ (σ is a CP equilibrium). Without loss of generality, we assume that $\Delta_k, \Delta_j > 0$, and that (y, r) and (y°, r°) are trading outcomes. In the game form following $c_{1,k,t} = \omega_{k,t} = 1$, referred to as game I, let $\tilde{r} = (\tilde{r}_b, \tilde{r}_s)$ ($\tilde{r}_b, \tilde{r}_s \in \{0,1\} \times [0,1]$) be the report specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ following the history where k adheres to $\sigma_{k,t}$ but j deviates to autarky. In the game form following $\omega_{k,t} = 0$, referred to as game II, let $\hat{r} = (\hat{r}_b, \hat{r}_s)$ ($\hat{r}_b, \hat{r}_s \in \{0,1\} \times [0,1]$) be the report specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ following the history where j adheres to $\sigma_{j,t}$ but k deviates to autarky. Let $\breve{r} = (\tilde{r}_b, \hat{r}_s)$. For σ to be an equilibrium, $v_k(\hat{r}) \leq v_k(r)$ (k does not deviate to autarky in game II) and $v_k(\breve{r}) \leq v_k(\hat{r})$ (k does not deviate to announce $\tilde{r}_b$ at autarky after autarky is reached). By the same argument (applied to j and game I and j's deviation to $\hat{r}_s$ at autarky), $v_j(\tilde{r}) \leq v_j(r^\circ)$ and $v_j(\breve{r}) \leq v_j(\tilde{r})$. So $(0, \breve{r})$ is dominated by $(0, r^\circ)$ and $(0, r)$ ($\Delta_k, \Delta_j > 0$).

game as the break-down probability approaches 0 (cf. Osborne and Rubinstein [28, Ch 4.2]). Also, it is easy to generalize the analysis for equal weights on the buyer and seller to non equal weights.

Let $(0, r')$ maximize the Nash product with the disagreement point $(0, \breve{r})$ given there is no good remaining. (Without loss of generality we treat r' as a report while it may be a lottery over reports.) Because k can obtain $(0, \breve{r})$ given j does not deviate in game II, for σ to be a RP equilibrium, $v_k(r') \leq v_k(r)$ (k does not gain by renegotiation). By the same argument (applied to j and game I), $v_j(r') \leq v_j(r^\circ)$. But then $(0, r')$ is dominated by $(0, r^\circ)$ ($\Delta_k > 0$), a contradiction. ■

In the above proof, $\breve{r}$ is constructed to deal with the concern that following different plays reaching autarky, the reports specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ need not all have the same continuation values and need not all be inefficient. The key to Proposition 3 is that renegotiation enforces the on-path report following $\omega_{k,t} = c_{1,k,t} = 1$ equivalent to the on-path report following $\omega_{k,t} = 0$ in continuation values. CP does not impose such equivalence, which is critical for why in the next section there is a folk theorem with CP equilibrium but not with RP equilibrium when $\rho < 1$.

The next result is parallel to Proposition 2. Cast into the above one-shot game, it pertains to the reward and punishment capacities of the third party, necessary and sufficient for y^* to be transferred in a RP equilibrium; again, sufficiency is not established here for it arises naturally below. Effectively, this result also implies the greatest lower bounds on $v_k(r) - v_k(r^*)$ and $v_j(r^*) - v_j(r)$ in Proposition 3.

For this result, we first identify the space of some relevant reports with an interval. Then RP implies that $v_k(.)$ and $v_j(.)$ are continuous and monotonic and satisfy a functional inequality. The main challenging problem is to find from $v_k(.)$ and $v_j(.)$ with these properties the pair such that $v_k(.)$ has the minimal increment over the interval given the increment of $v_j(.)$. To solve the problem, we develop a technique built on theory of integral equation. The result is centered around the function $x \mapsto \varphi(x, y, l, \delta)$ on $[0, y]$ defined by

$$\varphi(x, y, l, \delta) = -\frac{\alpha(\delta)}{l - \alpha(\delta)y} \int_0^x \frac{\alpha(\delta)u(\tau)}{\exp[\frac{\alpha(\delta)}{l-\alpha(\delta)y}(x - \tau)]} d\tau \qquad (2)$$

for $l > \alpha(\delta)y > 0$. Formally we have the following.

Proposition 4 *Let σ be a RP equilibrium. Fix t, $(\theta_i^t, \phi_i^t, r_i^{t-1})_{i \in I}$, and k with $\theta_{k,t} = 0$. Let j, (y^*, r^*), and $v_i(.)$ be given by Proposition 2. Let $l_k^* = \sup_r [v_k(r) - v_k(r^*)]$ and $l_j^* = \sup_r [v_j(r^*) - v_j(r)]$. Then $l_k^* \geq \varphi_b(y^*, l_j^*, \delta)$*

and $l_j^* \geq \varphi_s(y^*, \delta)$. *Here* $\varphi_s(y,\delta) = \min\{l > \alpha(\delta)y : \varphi(x,y,l,\delta)$ *is concave in* $x\}$ *for* $y > 0$, *and* $\varphi_b(y,l,\delta) = -\varphi(y,y,l,\delta)$ *for* $l > \alpha(\delta)y > 0$; $\varphi_s(y,\delta)$ *is defined,* $\varphi(x,y,l,\delta)$ *is concave in* x *iff* $l \geq \varphi_s(y,\delta)$, *and* $\varphi_s(y,\delta) \in (0.5\alpha(\delta)\nu(y), \alpha(\delta)\nu(y))$ *with* $\nu(y) = u(y)/u'(y) + y$.

Proof. We outline the proof here and verify some intermediate results in the appendix. The starting point is the stage-2 game form following j consuming some $c \in [1-y^*, 1)$ at stage 1 when $\omega_{k,t} = 1$. Denote by (x, r_x) the outcome of this game form specified by $\sigma_{k,t}$ and $\sigma_{j,t}$. Result 1 (verified in the appendix) says that $x = 1 - c$, and there exists a report $\breve{r}_x$ such that $(0, \breve{r}_x)$ is Pareto dominated by (x, r_x) and each of k and j can obtain $(0, \breve{r}_x)$ given the other does not deviate. (Here $(0, \breve{r}_x)$ has the similar role as $(0, \breve{r})$ in the proof of Proposition 3.) Then for σ to be a RP equilibrium, (x, r_x) must maximize the Nash product with the disagreement point $(0, \breve{r}_x)$ given $1 - c$ units of good remaining and, in particular, it cannot be dominated by the transfer x and a lottery over r_x and an arbitrary report r. So letting

$$
\begin{aligned}
G_k(z; r, x) &= \alpha(\delta)u(x) + z[v_k(r) - v_k(r_x)] + [v_k(r_x) - v_k(\breve{r}_x)], \\
G_j(z; r, x) &= -\alpha(\delta)x + z[v_j(r) - v_j(r_x)] + [v_j(r_x) - v_j(\breve{r}_x)],
\end{aligned}
$$

we have $G_k(0; r, x) > 0$, $G_j(0; r, x) > 0$, and

$$
0 \in \arg\max_{z \in [0,1]} G_k(z; r, x)^{1/2} G_j(z; r, x)^{1/2}. \tag{3}
$$

Next we treat c as a variable over $[1-y^*, 1)$ and obtain two functions $x \mapsto v(x) \equiv v_k(r_x)$ and $x \mapsto w(x) \equiv v_j(r_x)$ on $(0, y^*]$. Result 2 says that $-v(.)$ and $w(.)$ are nondecreasing and continuous on $(0, y^*]$, and that $v(0) - v(y^*) < \alpha(\delta)u(y^*)$ and $w(y^*) - w(0) \geq \alpha(\delta)y^*$ with $(v(0), w(0)) = \lim_{x \downarrow 0}(v(x), w(x))$. (Monotonicity here is stronger than Proposition 2 which only deals with y^* and y°.) For $x \in [0, y^*]$, define

$$
\begin{aligned}
\hat{v}(x) &= v(x) - [v_k(r^*) + l_k^*], \qquad (4) \\
\hat{w}(x) &= w(x) - [v_j(r^*) - l_j^*]. \qquad (5)
\end{aligned}
$$

Because $v_k(r^*) + l_k^* \geq v_k(\breve{r}_x)$ and $v_j(\breve{r}_x) \geq v_j(r^*) - l_j^*$ (by definition of l_k^* and l_j^*), in (3) setting $r = r_{x'}$ and using $\hat{w}(x) - \alpha(\delta)x > 0$ (by $G_j(0; r, x) > 0$), we have

$$
\hat{v}(x) - \hat{v}(x') \geq \frac{\alpha(\delta)u(x) + \hat{v}(x)}{\hat{w}(x) - \alpha(\delta)x}[\hat{w}(x') - \hat{w}(x)], \; 0 < x < x' \leq y^*. \tag{6}
$$

Next from (6) we derive $(\hat{v}(.), \hat{w}(.))$ with $(-\hat{v}(0), \hat{w}(0)) \geq (0,0)$ and the properties inherited from result 2 that minimize $\hat{v}(0) - \hat{v}(y^*)$. We first consider continuously differentiable $\hat{v}(.)$ and $\hat{w}(.)$ with $\hat{w}(0) > 0$ (so only the basic calculus is used). Let

$$\Delta(x) = -\hat{v}'(x) + [\alpha(\delta)u(x) + \hat{v}(x)]h(x), \tag{7}$$

$$h(x) = \frac{-\hat{w}'(x)}{\hat{w}(x) - \alpha(\delta)x}. \tag{8}$$

From (7) (and $\hat{v}(x) - \hat{v}(0) = \int_0^x \hat{v}'(\tau)d\tau$) we have

$$\hat{v}(x) = [\hat{v}(0) + \alpha(\delta)\int_0^x h(\tau)u(\tau)d\tau - \int_0^x \Delta(\tau)d\tau] + \int_0^x h(\tau)\hat{v}(\tau)d\tau. \tag{9}$$

Taking $\Delta(.)$ and $h(.)$ and $\hat{v}(0)$ as given, (9) defines an integral equation in $\hat{v}(.)$ which has a unique solution $\hat{v}(x) = \hat{v}_1(x) + \hat{v}_2(x)$ with

$$\hat{v}_n(x) = m_n(x) + \int_0^x m_n(\tau)h(\tau)K(\tau;x)d\tau, \tag{10}$$

where $m_1(x) = \alpha(\delta)\int_0^x h(\tau)u(\tau)d\tau$, $m_2(x) = \hat{v}(0) - \int_0^x \Delta(\tau)d\tau$, and $K(\tau;x) = \exp[\int_\tau^x h(\zeta)d\zeta]$ (cf. [30, Ch 9.1-1, p.459; Ch 2.9-1.2, p.171]). Result 3 says that

$$\hat{v}_1(x) = \alpha(\delta)\int_0^x u(\tau)K(\tau;x)h(\tau)d\tau, \tag{11}$$

$$\hat{v}_2(x) = \hat{v}(0)K(0;x) - \int_0^x K(\tau;x)\Delta(\tau)d\tau. \tag{12}$$

By (11) and $h(\tau) \leq 0$ ($\hat{w}'(\tau) \geq 0$ and $\hat{w}(\tau) - \alpha(\delta)\tau > 0$), $\hat{v}_1(0) - \hat{v}_1(y^*) = -\hat{v}_1(y^*) \geq 0$. Result 3 also says that $\hat{v}_1(x)$ can be written as

$$\hat{v}_1(x) = -\alpha(\delta)u(x) + \frac{\alpha(\delta)}{\hat{w}(x) - \alpha(\delta)x}\int_0^x \frac{[\hat{w}(\tau) - \alpha(\delta)\tau]u'(\tau)}{\exp[\int_\tau^x \frac{\alpha(\delta)}{\hat{w}(\zeta) - \alpha(\delta)\zeta}d\zeta]}d\tau. \tag{13}$$

Then by (5), $w(\tau) - \alpha(\delta)\tau \leq v_j(r^*) - \alpha(\delta)y^*$ (Proposition 2), and continuity of $\hat{w}(.)$, we see from (13) that $\hat{v}_1(0) - \hat{v}_1(y^*)$ is minimized iff $\hat{w}(\tau) - \alpha(\delta)\tau = l_j^* - \alpha(\delta)y^*$, all τ. By (12) and $\Delta(\tau) \geq 0$ (use (6)), $v_2(0) - \hat{v}_2(y^*) = \int_0^{y^*} K(\tau;y^*)\Delta(\tau)d\tau \geq 0$ is minimized iff $\Delta(\tau) = 0$, all τ. With these data, (11) and (12) imply that $\hat{v}(0) - \hat{v}(y^*)$ is minimized iff $\hat{v}(x) - \hat{v}(0) = \varphi(x, y^*, l_j^*, \delta)$ and $\hat{w}(x) - \alpha(\delta)x = l_j^* - \alpha(\delta)y^*$, all x.

We then move to adapt the above analysis for general $\hat{v}(.)$ and $\hat{w}(.)$ with $\hat{w}(0) > 0$. As Result 4, $\hat{v}(0) - \hat{v}(y^*)$ is minimized iff $\hat{v}(.)$ and $\hat{w}(.)$ take the same forms as above. For this result, we first apply the Lebesgue Theorem and Lebesgue decomposition (cf. Billingsley [4, Theorem 31.2, p.404; Eq. 31.31, p.414]) to $\hat{v}(.)$ and $\hat{w}(.)$. Then we show that (9) holds if the singular component of $\hat{v}(x)$ is added to its right side. Then we show that the modified (9) has a unique solution in $\hat{v}(.)$. For the remaining analysis, a key is to replace $\hat{w}(\tau)$ with the sum of its absolutely continuous component and $\hat{w}(y^*)$ at certain places so that integration can go through in a desirable way.

Result 4 helps us to establish Result 5: $\hat{v}(0) - \hat{v}(y^*) > -\varphi(y^*, y^*, l_j^*, \delta)$ if $\hat{w}(0) = 0$. We conclude that Result 4 holds without the restriction on $\hat{w}(0)$. Now by definition, $l_k^* \geq \hat{v}(0) - \hat{v}(y^*)$ so $l_k^* \geq \varphi_b(y^*, l_j^*, \delta)$. To obtain the greatest lower bound on l_j^* (we already know $l_j^* > \alpha(\delta)y^*$ by $G_j(0; r, y^*) > 0$), let (2) be written as

$$\varphi(x, y, l, \delta) = -\alpha(\delta)u(x) + \alpha(\delta) \int_0^x \frac{u'(\tau)}{\exp[\frac{\alpha(\delta)}{l - \alpha(\delta)y}(x - \tau)]} d\tau \qquad (14)$$

(as (11) can be written as (13)). Next fix $\hat{v}(0) - \hat{v}(y^*) < \alpha(\delta)u(y^*)$. By (14), $\varphi_b(y, l, \delta)$ is decreasing in l. Therefore the minimal l_j^* satisfying $\hat{v}(0) - \hat{v}(y^*) \geq \varphi_b(y^*, l_j^*, \delta)$ is obtained as $\hat{v}(0) - \hat{v}(y^*) = \varphi_b(y^*, l_j^*, \delta)$. Fix l_j^* at this minimal value so $\hat{v}(x) - \hat{v}(0) = \varphi(x, y^*, l_j^*, \delta)$ and $\hat{w}(x) - \alpha(\delta)x = l_j^* - \alpha(\delta)y^*$, all x. By the linearity of $\hat{w}(.)$ and efficiency of (x, r_x) in the game form following $c = 1 - x$, $\varphi(x, y^*, l_j^*, \delta)$ is concave in x. Result 6 says that $\varphi_s(y, \delta)$ and $\varphi(x, y, l, \delta)$ have properties given in the proposition. This completes the proof. ■

In the context of the above one-shot game, the proof of the last proposition characterizes the reward scheme that is efficient according to the third party's criteria, i.e., minimizing his payoff capacities to reward and punish. Notably, with such a scheme, as the seller's consumption c varies from $1 - y^*$ to approaching 1, the seller's surplus is positive and constant and his value function for corresponding reports is affine, and the buyer's surplus is positive and decreasing to zero and his value function is strictly concave; at $c = 1$ each agent gets zero surplus (so the seller's surplus and value function are not continuous but the buyer's are).

5 The folk (and non-folk) theorems

In this section, our main focus is how the refinements in the last section would restrict on the equilibrium allocations when people are patient. The benchmark is the following folk theorem with Definition 3 equilibrium.

Proposition 5 *Let* $\bar{w} = 0.5\rho[u(q^*) + (1 - q^*)]$ *(see* (1)*) and* $\underline{w} = 0.5\rho$*. Fix* $w \in (\underline{w}, \bar{w}]$*. Let* q *satisfy* $0.5\rho[u(q) + (1 - q)] = w$ *and* $\underline{\delta}$ *satisfy* $w - \underline{w} = \alpha(\underline{\delta})q$*. Let the allocation* f *be such that for any* $(\theta_i^t, \phi_i^t, \omega_i^{t-1})_{i \in I}$ *and* k*,* $f_{k,t}((\theta_i^t, \phi_i^t, \omega_i^{t-1})_{i \in I}, \omega_{k,t}) = q$ *if* $\omega_{k,t} = 1$*. Then* f *is supported by an equilibrium (Definition* 3*) if and only if* $\delta \geq \underline{\delta}$*.*

Proof. The "only-if" part is obvious. (Suppose the converse. For an on-path seller, if he transfers q, his payoff is $(1 - \delta)(1 - q) + \delta w$. If he transfers 0, his payoff cannot be lower than $(1 - \delta) \cdot 1 + \delta \underline{w}$.) For the "if" part, by Proposition 1, it suffices to show that f can be supported by an equilibrium in the perfect-monitoring benchmark. But this is standard. ■

5.1 CP equilibrium

In this part we show that the f in Proposition 5 can be supported by a CP equilibrium if and only if $\delta \geq \underline{\delta}$. While the strengthened equilibrium does not restrict the folk theorem (compared with Proposition 5), it does restrict the equilibrium strategy profiles that support f.

To proceed, we introduce some objects used in constructing equilibria in the rest of this section. In an equilibrium, consider a date-t meeting in which the buyer's and seller's start-of-t continuation values are v_b and v_s, respectively. Let $\varsigma = (v_b, v_s)$. When the seller is not endowed, let $b(\varsigma)$ and $s(\varsigma)$ be the buyer's and seller's start-of-t+1 values (implied by the equilibrium), respectively. When the seller is endowed, let $y(\varsigma)$, $\underline{v}_b(\varsigma)$, and $\bar{v}_s(\varsigma)$ be the transfer of the good, the buyer's start-of-t+1 value, and the seller's start-of-t+1 value, respectively. Also, when the seller is endowed but consumes 1 at stage 1, let $\bar{v}_b(\varsigma)$ and $\underline{v}_s(\varsigma)$ be the buyer's and seller's start-of-t+1 values, respectively. Let $A(\varsigma) = (y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma), \bar{v}_b(\varsigma), \underline{v}_s(\varsigma))$.

Now let σ be a CP equilibrium. Suppose that in a meeting when the seller is endowed, by some way (discussed below) a report is identified with some $z \in [0, 1]$, the zero transfer is revealed by a report identified with 0, and the buyer's and seller's continuation values implied by a report identified with z are expressed as $\bar{v}_b(\varsigma) + \iota_b(z,\varsigma)$ and $\underline{v}_s(\varsigma) + \iota_s(z,\varsigma)$, respectively. If $\iota_b(0,\varsigma) = \iota_s(0,\varsigma) = 0$, then

$$U_b(y,z,\varsigma) = (1-\delta)u(y) + \delta\iota_b(z,\varsigma), \quad U_s(y,z,\varsigma) = -(1-\delta)y + \delta\iota_s(z,\varsigma), \tag{15}$$

are the buyer's and seller's surplus from a transfer y and a report identified with z, respectively. If in addition $\iota_b(z,\varsigma)$ and $\iota_b(z,\varsigma)$ are concave in z, then

$$(\acute{y}(c,\varsigma), \acute{z}(c,\varsigma)) \in \arg \max_{(y,z)\in[0,1-c]\times[0,1]} U_b(y,z,\varsigma) \ s.t. \ U_s(y,z,\varsigma) \geq U_s(0,0,\varsigma) \tag{16}$$

is a pairwise efficient outcome following $c_{1,k,t} = c$. Suppose further that σ supports the f in Proposition 5 and let one agent's start-of-t continuation value be v. Because his partner's start-of-t value is w with probability one,

$$\begin{aligned} v/\delta \ = \ & 0.5\rho[\alpha(\delta)u(y(v,w)) + \underline{v}_b(v,w)] + 0.5(1-\rho)b(v,w) \\ & +0.5\rho[\alpha(\delta)(1-y(w,v)) + \bar{v}_s(w,v)] + 0.5(1-\rho)s(w,v). \end{aligned} \tag{17}$$

Notice that if $v = w$ then $y(w,w) = q$ and $\underline{v}_b(w,w) = \bar{v}_s(w,w) = b(w,w) = s(w,w) = w$, and, hence, the right side of (17) is equal to w. But can (17) hold for any $v \neq w$?[17] And, is the endowed seller willing to consume no more than $1 - y(\varsigma)$ at stage 1 if the outcome following his stage-1 consumption c is determined by (16)? The next lemma shows that these are possible.

Lemma 1 *Let $(w,q,f,\underline{\delta})$ be given by Proposition 5. Let $\delta \geq \underline{\delta}$, $\underline{v} = w - \alpha(\delta)q$, $w < \bar{v} \leq w + \max\{\rho q(1-\delta)/(2-\delta), \alpha(\delta)u'(q)q\}$, and $V = [\underline{v}, \bar{v}]$. Then there exist $A(.)$, $b(.)$, and $s(.)$ such that (17) holds for $v \in V$. For $\varsigma = (v_b, v_s) \in V \times V$, given $A(\varsigma)$ there exist concave $\iota_b(.,\varsigma)$ and $\iota_s(.,\varsigma)$ with $\iota_b(0,\varsigma) = \iota_s(0,\varsigma) = 0$ such that $(\acute{y}(c,\varsigma), \acute{z}(c,\varsigma))$ (defined by (15) and (16)) can be set as $(1-c, 1-c)$ for $c \geq 1 - y(\varsigma)$ and as $(y(\varsigma), y(\varsigma))$ for $c < 1 - y(\varsigma)$, and $1 - y(\varsigma) \in \arg\max_{c\in[0,1]} U_s(\acute{y}(c,\varsigma), \acute{z}(c,\varsigma), \varsigma)$.*

Proof. Fix $\varsigma = (v_b, v_s)$. First we construct $(y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma))$ in five exclusive and exhaustive cases.

(a) $(v_b - w)(v_s - w) \neq 0$. Then $(y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma)) = (0, v_b, v_s)$. (This case deals with a meeting between two agents with off-path continuation values.)

(b) $(v_b, v_s) = (w, v)$ and $v \geq w$. Then $(\underline{v}_b(\varsigma), \bar{v}_s(\varsigma)) = (w, v)$ and

$$y(\varsigma) = q - \frac{(v-w)(2-\delta)}{\rho(1-\delta)}.$$

[17] Proposition 2 requires that $\bar{v}_s(w,w) - \underline{v}_s(w,w) \geq \alpha(\delta)q$ and $\bar{v}_b(w,w) - \underline{v}_b(w,w) > 0$.

(By the upper bound on $\bar{v}$, $y(\varsigma) \geq 0$. This case shows the basic idea of how to reward an agent with $v > w$ (compared to an agent with w): when such an agent makes a transfer less than q as a seller to an agent with w, his own future value is still v.)

(c) $(v_b, v_s) = (v, w)$ and $v \geq w$. Then $(y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma)) = (q, w, w)$.

(d) $(v_b, v_s) = (w, v)$ and $v < w$. Then $(y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma)) = (0, w, J(v))$,

$$J(v) = v \cdot \min\{\frac{w}{v}, 1 + \alpha(\delta)(1 - \frac{\underline{w}}{v})\}. \tag{18}$$

In (18), $\underline{w}$ is given by Proposition 5. (With $\bar{v}_s(w, v)$ here and $\underline{v}_b(v, w)$ in case (e) equated to $J(v)$, an agent with v returns to w in finite periods unless $v = \underline{v} = \underline{w}$.)

(e) $(v_b, v_s) = (v, w)$ and $v < w$. Then $(\underline{v}_b(\varsigma), \bar{v}_s(\varsigma)) = (J(v), w)$ and

$$u(y(\varsigma)) = u(q) - q - \frac{w}{0.5\rho} + \frac{v - \delta J(v)}{0.5\rho(1-\delta)}.$$

(By definitions of $J(v)$, w, and $\underline{w}$, $0.5\rho u(y(\varsigma)) = v + [v - J(v)]\delta/(1-\delta) - \underline{w} \geq 0$ so $y(\varsigma) \geq 0$; by $v \leq J(v) \leq w$, $u(y(\varsigma)) < u(q)$ or $y(\varsigma) < q$. This case shows how to punish an agent with $v < w$: when such an agent receives a transfer less than q as a buyer from an agent with w, his meeting partner's future value is still w.)

Next let $(b(\varsigma), s(\varsigma)) = (\underline{v}_b(\varsigma), \bar{v}_s(\varsigma))$. Then (17) holds for $v \in V$. (To verify, substitute the above constructed terms for a generic v into the right side of (17) and then subtract w from the left side and the above terms for $v = w$ from the right side.) Also, let $\bar{v}_b(\varsigma) = \underline{v}_b(\varsigma) + (\bar{v} - w)y(\varsigma)/q$ and $\underline{v}_s(\varsigma) = \bar{v}_s(\varsigma) - \alpha(\delta)y(\varsigma)$.

Next let

$$\iota_b(z, \varsigma) = -\min\{z, y(\varsigma)\} \cdot (\bar{v} - w)/q, \tag{19}$$

$$\iota_s(z, \varsigma) = \bar{v}_s(\varsigma) - \underline{v}_s(\varsigma) + \alpha(\delta) \cdot \min\{z - y(\varsigma), 0\}. \tag{20}$$

Then by $y(\varsigma) \leq q$ and $\alpha(\delta)u'(q)q \geq \bar{v} - w$, $(1-c, 1-c)$ and $(y(\varsigma), y(\varsigma))$ solve the problem in (16) for $c \geq 1 - y(\varsigma)$ and for $c < 1 - y(\varsigma)$, respectively. It follows that $U_s(\acute{y}(c, \varsigma), \acute{z}(c, \varsigma), \varsigma) = 0$, all c. This completes the proof. ■

Now we can present the following folk theorem for CP equilibrium.

Proposition 6 *Let $(w, q, f, \underline{\delta})$ be given by Proposition 5. Then f is supported by a CP equilibrium if and only if $\delta \geq \underline{\delta}$.*

Proof. The "only-if" part follows from Proposition 5. For the "if" part, let V and $(A(.), b(.), s(.), \iota_b(.), \iota_s(.))$ be given by Lemma 1 and its proof. We proceed by three steps. Step 1 defines a mapping $h_{k,t}$ (for any k and t) which maps all agents' public history available at the start of t, i.e., $\beta_{t-1} \equiv (\theta_i^{t-1}, \phi_i^{t-1}, r_i^{t-1})_{i\in I}$, into an element in V (which turns out to be k's start-of-t continuation value). Step 2 describes the mechanism T and strategy profile σ. Step 3 verifies that σ is a CP equilibrium.

Step 1. Let $h_{k,1}(\beta_0) = w$ and then define $h_{k,t}$ for $t \geq 2$ by induction. Fix β_{t-1} and let $j = \phi_{k,t}$. Given β_{t-1}, let $v_{k,t} = g_b(r_{k,t-1}, h_{k,t-1}(\beta_{t-2}), h_{j,t-1}(\beta_{t-2}))$ if $\theta_{k,t} = 0$ and $v_{k,t} = g_s(r_{k,t-1}, h_{j,t-1}(\beta_{t-2}), h_{k,t-1}(\beta_{t-2}))$ if $\theta_{k,t} = 1$. For $\varsigma = (v_b, v_s)$ we define $g(r, \varsigma) = (g_b(r, \varsigma), g_s(r, \varsigma))$ as follows.

If $r \neq \mathbf{0} \equiv (0, 0, 0, 0)$ and $r_b^1 r_s^1 = 0$, then $g(r, \varsigma) = (\underline{v}, \underline{v})$. (This is to make the autarky outcomes the worst outcome.)

If $r = \mathbf{0}$, then $g(r, \varsigma) = (\underline{v}_b(\varsigma), \bar{v}_s(\varsigma))$. (This is to pin down the on-path report when the seller is not endowed; recall that $(b(\varsigma), s(\varsigma)) = (\underline{v}_b(\varsigma), \bar{v}_s(\varsigma))$.)

If $r_b^1 r_s^1 = 1$, then $g(r, \varsigma) = (\bar{v}_b(\varsigma) + \iota_b(r_b^2, \varsigma), \underline{v}_s(s) + \iota_s(r_b^2, \varsigma))$. (This is to make (19)-(20) relevant value functions when the seller is endowed.)

Let $D_t = \{i \in I : v_{i,t} < w\}$ (this set turns out to be the set of reported defectors at the start of t and it is defined because of k's representative role). Then let $h_{k,t}(\beta_{t-1})$ be determined as follows.

(i) $\#D_t \leq 1$, i.e., the number of elements in D_t is no greater than one (there is at most one defector). Then $h_{k,t}(\beta_{t-1}) = v_{k,t}$.

(ii) $D_t = \{i_1, i_2\}$, $v_{i_1,t-1} < w$ and $v_{i_2,t-1} = w$ (there are an old defector, i_1, and a new defector, i_2). If $k \notin \{i_2, \phi_{i_2,t-1}\}$, then $h_{k,t}(\beta_{t-1}) = w$; otherwise $h_{k,t}(\beta_{t-1}) = v_{k,t}$.

(iii) Other D_t. Then $h_{k,t}(\beta_{t-1}) = w$.

Step 2. Fix $\gamma_t \equiv (\theta_i^t, \phi_i^t, r_i^{t-1})_{i\in I}$, k, and $\omega_{k,t}$. Given γ_t, let $v_i = h_{i,t}(\beta_{t-1})$, $i \in \{k, j = \phi_{k,t}\}$. Let $\varsigma = (v_k, v_j)$ if $\theta_{k,t} = 0$ and $\varsigma = (v_j, v_k)$ if $\theta_{k,t} = 1$. In the game form prescribed by $T_{k,t}$ (following $c_{1,k,t}$), k and j simultaneously propose an outcome. If k and j propose the same feasible (y, r), then (y, r) is the outcome; otherwise autarky is reached. Actions specified by $\sigma_{k,t}$ are as follows (we only describe $\sigma_{k,t}$ because of k's representative role).

Stage 1. If $\theta_{k,t} = 1$, choose $c_{1,k,t} = \min\{\omega_{k,t}, 1 - y(\varsigma)\}$.

Stage 2. If $\omega_{k,t} = 0$, propose $(0, \mathbf{0})$. If $(\omega_{k,t}, c_{1,k,t}) = (1, c)$, propose $(\acute{y}(c, \varsigma), \acute{r}(c, \varsigma)))$ with $\acute{r}(c, \varsigma) = (1, \acute{z}(c, \varsigma), 1, \acute{z}(c, \varsigma))$, where $(\acute{y}(c, \varsigma), \acute{z}(c, \varsigma))$ is given by Lemma 1. In autarky, announce $(0, 1)$.

Step 3. Fix γ_t, k, and $\omega_{k,t}$. Let j, v_i, and ς be the same as in step 2. We claim (a) If all agents follow σ from t on, $v_j = w$, and k's start-of-t+1 continuation value is determined by $h_{k,t+1}$, then k's start-of-t continuation value is v_k. For this claim, by its hypotheses and Lemma 1, k's payoff from meeting j is $(1-\delta)u(y(v_k,w))+\delta\underline{v}_b(v_k,w)$ if $(\theta_{k,t},\omega_{k,t})=(0,1)$, is $(1-\delta)[1-y(w,v_k)]+\delta\bar{v}_s(w,v_k)$ if $(\theta_{k,t},\omega_{k,t})=(1,1)$, is $\delta b(v_k,w)$ if $(\theta_{k,t},\omega_{k,t})=(0,0)$, and is $\delta s(w,v_k)$ if $(\theta_{k,t},\omega_{k,t})=(1,0)$. Because (17) holds with $(A(.),b(.),s(.))$, v_k is k's start-of-t value.

Given (a), we claim (b) If any $i\neq k$ follows σ_i from t on and k follows σ_k from $t+1$ on, then k does not gain by a single deviation from $\sigma_{k,t}$. This claim is obvious for the move in autarky (notice that $g(r,\varsigma)=(\underline{v},\underline{v})$ if $r\neq\mathbf{0}$ and $r_b^1r_s^1=0$), for any non-autarky move when $\omega_{k,t}=0$ or $(\theta_{k,t},\omega_{k,t})=(0,1)$, and for the stage-2 proposal when $(\theta_{k,t},\omega_{k,t})=(1,1)$ following $c_{1,k,t}$. When $(\theta_{k,t},\omega_{k,t},c_{1,k,t})=(1,1,c)$, k's surplus from the outcome specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ is $U_s(\acute{y}(c,\varsigma),\acute{z}(c,\varsigma),\varsigma)$; hence by Lemma 1, k does not gain by choosing $c\neq 1-y(\varsigma)$ at stage 1.

Given (a) and (b), σ is an equilibrium. To see that σ is a CP equilibrium, suppose $\acute{y}(c,\varsigma)>0$ (treating $\acute{y}(c,\varsigma)=0$ as a special case). We observe that $z'\mapsto g((1,z,1,z'),\varsigma)$ is constant, $z\mapsto g((1,z,1,z'),\varsigma)$ is affine over $[0,y(\varsigma)]$ and constant over $[y(\varsigma),1]$, and $g((1,y(\varsigma),1,z'),\varsigma)=g(\mathbf{0},\varsigma)>g(r,\varsigma)=(\underline{v},\underline{v})$ if $r\neq\mathbf{0}$ and $r_b^1r_s^1=0$. So $(0,\mathbf{0})$ is pairwise efficient following $\omega_{k,t}=0$. By Lemma 1, $(\acute{y}(c,\varsigma),\acute{r}(c,\varsigma))$ maximizes $\alpha(\delta)u(y)+g_b(r,\varsigma)-g_b((1,0,1,0),\varsigma)$ subject to $g_s(r,\varsigma)-\alpha(\delta)y\geq g_s((1,0,1,0),\varsigma)$ and $0\leq y\leq 1-c$, and, hence, by the above observation, it is pairwise efficient following $(\omega_{k,t},c_{1,k,t})=(1,c)$. This completes the proof. ■

5.2 RP equilibrium: deterministic endowment

In this part we show that when $\rho=1$ the f in Proposition 5 can be supported by a RP equilibrium but the greatest lower bound on the discount factor δ to support f is greater than $\underline{\delta}$. Compared with Proposition 6, necessity of a higher lower bound on δ represents a loss due to renegotiation.

To see why a higher lower bound on δ is necessary, suppose that σ in Proposition 4 supports f. Let k and j's start-of-t continuation values be w. Set $(v_k(r^*),v_j(r^*),y^*)=(w,w,q)$. By the proposition, $l_j^*\geq\varphi_s(q,\delta)>0.5\alpha(\delta)\nu(q)$. But this cannot hold for δ close to $\underline{\delta}$ because $0.5\nu(q)>q$ and $l_j^*\leq w-\underline{w}=\alpha(\underline{\delta})q$. Hence δ should at least satisfy $w-\underline{w}=\varphi_s(q,\delta)$ (it is shown in the appendix that $\varphi_s(y,\delta)$ and $\varphi_b(y,l,\delta)$ are decreasing in δ). To determine what the lower bound on δ is, we observe that if $\bar{v}$ is the largest continuation value admitted by σ, then for q to be transferred in the meeting in consideration, $l_k^* \leq \bar{v} - w$ so $\bar{v} \geq w + \varphi_b(q, w - \underline{w}, \delta)$.

Is it possible for $\bar{v}$ to reach this level for a given δ? To address this issue, let k's start-of-t value be $\bar{v}$ while keep j's as w. Let $y \in [0,1]$ be k's consumption and η be his start-of-t+1 value when he is the buyer; and let 1 be k's consumption and $\bar{v}$ be his start-of-t+1 value when he is the seller (this is the maximum k can get as a seller). Hence $\bar{v} = 0.5[(1-\delta)u(y) + \delta\eta] + 0.5[(1-\delta) + \delta\bar{v}]$, or $\bar{v} = \kappa(y, \eta, \delta)$, where

$$\kappa(y, \eta, \delta) = \psi(y) - \frac{0.5\delta}{1 - 0.5\delta}[\psi(y) - \eta], \ \psi(y) = 0.5[u(y) + 1].$$

By Proposition 4, for (y, η) to be realized in the meeting when k is the buyer, there must be a pair of (l_b, l_s) in

$$\Gamma(\delta) = \{(y, \eta, l_b, l_s) : 0 \leq y \leq 1, \ \underline{w} \leq \eta \leq \psi(y), \tag{21}$$
$$\kappa(y, \eta, \delta) - \underline{w} \geq l_s \geq \varphi_s(y, \delta), \ \kappa(y, \eta, \delta) - \eta \geq l_b \geq \varphi_b(y, l_s, \delta)\}$$

with $\varphi_s(0, \delta) = \varphi_b(0, l, \delta) = 0$. (As noted above, when both k and j have the on-path continuation value w, in absolute value the maximal punishment to the seller j is $w - \underline{w}$ and the maximal reward to the buyer k is $\kappa(y, \eta, \delta) - w$. Compared to these values, (21) says that there is more freedom to provide incentives when k has an off-path continuation value.) Then, an upper bound on $\bar{v}$ is

$$\bar{v}(\delta) = \max \kappa(y, \eta, \delta) \ s.t. \ (y, \eta, l_b, l_s) \in \Gamma(\delta). \tag{22}$$

Apparently, $\bar{v}(\delta)$ is defined. The next lemma gives useful properties of $\bar{v}(\delta)$.

Lemma 2 *Let $\rho = 1$. Let $(w, q, f, \underline{\delta}, \underline{w})$ and $\bar{v}(\delta)$ be given by Proposition 5 and (22), respectively. (i) $\hat{\delta} = \min\{\delta > \underline{\delta} : \bar{v}(\delta) - w \geq \varphi_b(q, w - \underline{w}, \delta)$, $w - \underline{w} \geq \varphi_s(q, \delta)\}$ is defined; and (ii) If f is supported by a RP equilibrium σ, then $\sup V \leq \bar{v}(\delta)$, where V is the set of continuation values admitted by σ.*

Proof. See the appendix. ■

By Lemma 2 and Proposition 4, if some σ supports f then $\delta \geq \hat{\delta}$. But how to fulfill each $v \leq \bar{v}(\delta)$ in σ? Letting $U_b(.)$ and $U_s(.)$ be given by (15) and

$$(\acute{y}(c, \varsigma), \acute{z}(c, \varsigma)) \in \arg \max_{(y,z) \in [0,1-c] \times [0,1]} [U_b(y, z, \varsigma)]^{1/2} [U_s(y, z, \varsigma)]^{1/2}, \tag{23}$$

then we have the following result parallel to Lemma 1.

Lemma 3 *Let $\rho = 1$. Let $(w, q, f, \underline{w})$ and $\hat{\delta}$ be given by Proposition* 5 *and Lemma* 2*, respectively. Let $\delta \geq \hat{\delta}$, $\underline{v} = \underline{w}$, $\bar{v} = \bar{v}(\delta)$, and $V = [\underline{v}, \bar{v}]$. Then there exists $A(.)$ such that* (17) *holds for $v \in V$. For $\varsigma = (v_b, v_s) \in V \times V$, given $A(\varsigma)$ there exist concave $\iota_b(., \varsigma)$ and $\iota_s(., \varsigma)$ with $\iota_b(0, \varsigma) = \iota_s(0, \varsigma) = 0$ such that $(\acute{y}(c, \varsigma), \acute{z}(c, \varsigma))$ (defined by* (15) *and* (23)*) can be set as $(1 - c, 1 - c)$ for $c \geq 1 - y(\varsigma)$ and as $(y(\varsigma), y(\varsigma))$ for $c < 1 - y(\varsigma)$, and $1 - y(\varsigma) \in \arg\max_{c \in [0,1]} U_s(\acute{y}(c, \varsigma), \acute{z}(c, \varsigma), \varsigma)$.*

Proof. The construction for $(y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma))$ is adapted from the proof of Lemma 1. The construction of $(\bar{v}_b(\varsigma), \underline{v}_s(\varsigma))$ is built on Proposition 4; that is, $\bar{v}_b(\varsigma) - \underline{v}_b(\varsigma) = \varphi_b(y(\varsigma), \varphi_s(y(\varsigma), \delta), \delta)$ and $\bar{v}_s(\varsigma) - \underline{v}_s(\varsigma) = \varphi_s(y(\varsigma), \delta)$, leading to the obvious construction of $\iota_b(., \varsigma)$ and $\iota_s(., \varsigma)$. Details are in the appendix. ■

Now we can present the following folk theorem for RP equilibrium.

Proposition 7 *Let $\rho = 1$. Let (w, q, f) and $\hat{\delta}$ be given by Proposition* 5 *and Lemma* 2*, respectively. Then f is supported by a RP equilibrium if and only if $\delta \geq \hat{\delta}$.*

Proof. The "only-if" part follows from Lemma 2 and Proposition 4. For the "if" part, let V and $(A(.), \iota_b(.), \iota_s(.))$ be given by Lemma 3 and its proof (here $b(.)$ and $s(.)$ are irrelevant). Now apply the three-step proof of Proposition 6 with the following modifications. In step 1, let $g(r, \varsigma) = (\bar{v}_b(\varsigma) + \iota_b(r_b^2, \varsigma), \underline{v}_s(\varsigma) + \iota_s(r_b^2, \varsigma))$. (That is, the values of a report for the buyer and seller depend only on the buyer's message of the transfer.) In step 2, for $\sigma_{k,t}$ remove the part for $\omega_{k,t} = 0$; let $(\acute{y}(c, \varsigma), \acute{z}(c, \varsigma))$ in $(\acute{y}(c, \varsigma), \acute{r}(c, \varsigma))$ (when $(\omega_{k,t}, c_{1,k,t}) = (1, c)$) be given by Lemma 3; and let $(0, 0)$ be announced in autarky. In step 3, Lemma 3 assures that k does not gain by choosing $c_{1,k,t} \neq 1 - y(\varsigma)$ when $\theta_{k,t} = 1$; and σ is a RP equilibrium because $(\acute{y}(c, \varsigma), \acute{r}(c, \varsigma))$ maximizes $[\alpha(\delta)u(y) + g_b(r, \varsigma) - g_b(\mathbf{0}, \varsigma)]^{1/2}[g_s(r, \varsigma) - g_s(\mathbf{0}, \varsigma) - \alpha(\delta)y]^{1/2}$ over (y, r) with $0 \leq y \leq 1 - c$ ($\mathbf{0} = (0, 0, 0, 0)$), and $\iota_b(r_b^2, \varsigma)$ and $\iota_s(r_b^2, \varsigma)$ are concave in r_b^2. ■

5.3 RP equilibrium: stochastic endowment

In this part we establish two results for RP equilibrium when $\rho < 1$. First we show that there is no folk theorem and there is a loss due to renegotiation which does not vanish as δ approaches 1 (i.e., there is no approximate folk theorem). Then we show that when δ exceeds a lower bound, the welfare loss due to renegotiation vanishes as ρ approaches 1, which provides some continuity going from $\rho < 1$ to $\rho = 1$.

We begin with the first result.

Proposition 8 *Let $\rho < 1$. (i) The f in Proposition 5 is not supported by any RP equilibrium; and (ii) If σ is a RP equilibrium, then the average continuation value at the start of date 1 is bounded above by some $\bar{v}_\rho < \bar{w}_\rho$, where $\bar{w}_\rho$ is the $\bar{w}$ (a function of ρ) in Proposition 5.*

Proof. For part (i), suppose the converse. But then by Proposition 3, the distribution of the continuation values at the start of date 2 is non degenerate, a contradiction. For part (ii), the average continuation value at the start of date 1 cannot exceed the maximum value of the following problem which includes the second inequality in Proposition 3 into the objective function to constrain choices in meetings of odd dates (even dates are left out for simplicity), namely,

$$\max_{(\pi, y_1, y_2)} \frac{1-\delta}{1-\delta^2}\{\lambda(y_1) + \delta[\pi\lambda(y_2) + (1-\pi)(\lambda(y_2) - \delta^{-1}y_1)]\} \tag{24}$$

subject to $0 \le \pi \le 1 - 0.5(1-\rho)$, $0 \le y_1, y_2 \le 1$, and $\lambda(y) = 0.5\rho[u(y) - y]$. (One may read the problem in (24) as follows. If an agent at an odd t is a seller and not endowed, then he cannot be in the proportion π of meetings at $t+1$. And, if one agent is in one of the proportion π of meetings at $t+1$, then his expected period utility at t+1 before his type realization is $(1-\delta)\lambda(y_2)$; otherwise, his expected period utility is $(1-\delta)(\lambda(y_2) - \delta^{-1}y_1)$, where y_1 is the meeting transfer at t when sellers are endowed. A new two-date cycle restarts at $t+2$.) In this problem, it is optimal to choose $\pi = 1 - 0.5(1-\rho)$, $y_1 = y$ with $u'(y) = 1/\rho$ ($y = 0$ if $u'(0) < 1/\rho$ and $y = 1$ if $u'(1) > 1/\rho$), and $y_2 = q^*$; hence the maximum value $\bar{v}_{\rho,\delta} = [\lambda(y) - 0.5(1-\rho)y + \delta\lambda(q^*)]/(1+\delta)$. Apparently, $\lim_{\delta\to 1} \bar{v}_{\rho,\delta} < \bar{w}_\rho$. ■

Proposition 8 is a consequence of the above-noted property of Proposition 3 that resembles the incentive compatibility constraint in the centralized risk-sharing models with private information. Among those models, our model is more comparable to the one in Green [11] where a nearly efficient allocation can be implemented for patient agents (cf. Fudenberg, Levine, and Maskin [10]). The proof of Proposition 8 illustrates why with decentralization near efficiency cannot be a RP equilibrium in our model: the transfer is pairwise so that $\lambda(y)$ is bounded above by $\lambda(q^*) = \bar{w}_\rho$ and, in particular, it cannot be improved upon by using the resources from the proportion $1-\pi$ of meetings at $t+1$. It is of great interest to further explore implications of the Proposition-3 property and decentralization on the optimal allocation. For example, is there an immiserization result as in [11]? Non-degenerate distributions of the continuation values make the analysis much more difficult and beyond the scope of this paper.[18]

Next we turn to the second result. Here we construct an equilibrium in which the distribution of the continuation values has a support $\{v_n\}_{n=-N+1}^{2}$ and the mass of agents with v_n is π_n. The basic idea is as follows. The values v_2 and v_{-N+1} provide incentives for people with v_1 (as their start-of-t continuation values) to transfer q^* when they meet and sellers are endowed and, as π_1 turns out to be somewhat proportional to ρ, the welfare loss vanishes as ρ approaches 1. Specifically, in a date-t meeting between two agents with v_1, if the seller is endowed, then the transfer is q^* and each agent's start-of-t+1 value is v_1; otherwise the buyer's and seller's start-of-t+1 values are v_2 and v_{-N+1}, respectively.

If agent k's value is v_{-N+1} at the start of t, then he consumes his own endowment for N consecutive dates (including t). At the start of the nth date of this N-date period, his value is v_{-N+n}; at the start of date $t+N$, the first date following the N-date period, k's value returns to v_1. In the N-date period, if the start-of-τ value of $j = \phi_{k,\tau}$ is v_1 or v_2, then j's start-of-τ+1 value is v_1.

What if k's value is v_2 at the start of t? When the start-of-t value of $j = \phi_{k,t}$ is v_2, or j's value is v_1 and k is a seller, k and j's start-of-t+1 values are always v_1 (so there is no transfer). When j's value is v_1 and k is a buyer, k and j are treated as if they both have v_1.

The way that an agent's value is updated implies the following inflow and outflow of the mass for agents with v_n at the end of t. If $n = -N + 1$, then the inflow is $0.5\pi_1(1-\rho)(\pi_1+\pi_2)$ (an agent with v_1 contributes to the inflow if he is a non-endowed seller and his meeting partner is with v_1 or v_2) and outflow is π_{-N+1}. If $n = 2$, then the inflow is $0.5(1-\rho)\pi_1\pi_1$ (an agent with v_1 contributes to the inflow if he is a buyer and his meeting partner is with

[18] Such distributions also make monetary matching models hard to analyze; only with special assumptions distributions can be explicitly solved (e.g., Green and Zhou [12]). To determine the optimal allocation in our model, Proposition 4 may play a fundamental role.

v_1 and not endowed) and outflow is $\pi_2[1-0.5(1-\rho)\pi_1]$ (an agent with v_2 does not contribute to the outflow if he is a buyer and his meeting partner is with v_1 and not endowed). If $-N+2 \leq n \leq 0$, then the inflow is π_{n-1} and outflow is π_n.

Hence the vector $\{v_n\}_{n=-N+1}^{2}$ satisfies

$$v_n/\delta = 0.5\rho\alpha(\delta) + v_{n+1}, \quad -N+1 \leq n \leq 0, \tag{25}$$

$$\begin{aligned} v_1/\delta = & (1-\pi_1-\pi_2)\{0.5\rho\alpha(\delta)+v_1\} \\ & +\pi_1\{0.5\rho\alpha(\delta)[u(q^*)+1-q^*]+\rho v_1+0.5(1-\rho)(v_2+v_{-N+1})\} \\ & +\pi_2\{0.5\rho\alpha(\delta)(1-q^*)+\rho v_1+0.5(1-\rho)(v_1+v_{-N+1})\}, \end{aligned} \tag{26}$$

$$\begin{aligned} v_2/\delta = & (1-\pi_1)\{0.5\rho\alpha(\delta)+v_1\} \\ & +\pi_1\{0.5\rho\alpha(\delta)[u(q^*)+1]+\rho v_1+0.5(1-\rho)(v_2+v_1)\}; \end{aligned} \tag{27}$$

and the vector $\{\pi_n\}_{n=-N+1}^{2}$ satisfies

$$\textstyle\sum_{n=-N+1}^{2}\pi_n = 1, \tag{28}$$

$$0.5(1-\rho)\pi_1(\pi_1+\pi_2) = \pi_{-N+1}, \tag{29}$$

$$0.5(1-\rho)\pi_1\pi_1 = \pi_2[1-0.5(1-\rho)\pi_1], \tag{30}$$

$$\pi_n = \pi_{n-1}, \quad -N+2 \leq n \leq 0. \tag{31}$$

To ensure conditions in Proposition 4, we also require

$$\frac{v_1 - v_{-N+1}}{\alpha(\delta)} \geq \nu(q^*), \tag{32}$$

$$\frac{v_2 - v_1}{\alpha(\delta)} \geq u(q^*) - \int_0^{q^*} \frac{u'(\tau)}{\exp[\frac{1}{(v_1-v_{-N+1})/\alpha(\delta)-q^*}(q^*-\tau)]} d\tau. \tag{33}$$

The next proposition confirms existence of such a RP equilibrium.

Proposition 9 *If $\delta < 1$ exceeds a lower bound, then for ρ sufficiently close to 1 there exists a RP equilibrium in which the average continuation value at the start of date 1 is some $v(\rho)$ such that $\lim_{\rho\to 1} v(\rho) = 0.5[u(q^*)+1-q^*]$.*

Proof. To begin with, fix a sufficiently large N so that

$$N0.5[u(q^*)-q^*] > \nu(q^*), \tag{34}$$

$$0.5q^* > u(q^*) - \int_0^{q^*} \frac{u'(\tau)}{\exp[\frac{1}{N0.5[u(q^*)-q^*]-q^*}(q^*-\tau)]} d\tau. \tag{35}$$

Next, by (29)-(31), $\pi_n = \pi_2$ for $n \leq 0$. Then by (28), $\pi_2 = (1-\pi_1)/(N+1)$. This and (30) imply that π_1 is equal to

$$\pi_1(\rho) = \frac{-[1+0.5(1-\rho)] + \sqrt{[1+0.5(1-\rho)]^2 + 2(1-\rho)N}}{N(1-\rho)}.$$

Notice that $0 < \pi_1(\rho) < 1$. With $\pi_1 = \pi_1(\rho)$, (29)-(31) determine a unique vector $\{\pi_n\}$. Given the vector $\{\pi_n\}$, (25)-(27) determine a unique vector $\{v_n\}$ (the right sides of (25)-(27) define a mapping of $\{v_n\}$ that satisfies the Blackwell's sufficient conditions). Next, as is shown in the appendix, some manipulation on (25)-(27) leads to expressions of $(v_1 - v_{-N+1})/\alpha(\delta)$ and $(v_2 - v_1)/\alpha(\delta)$ (in (42) and (43) below) which, viewed as functions of (ρ, δ), are continuous in (ρ, δ) and as $(\rho, \delta) \to (1, 1)$ their limits are $N0.5[u(q^*) - q^*]$ and $0.5q^*$, respectively. By continuity and (34) and (35), (32) and (33) hold for ρ sufficiently close to 1 if $\delta < 1$ exceeds a lower bound.

Now fix (ρ, δ) in the suitable range. For the corresponding $\{(v_n, \pi_n)\}$, let $V = [v_{-N+1}, v_2]$ and we construct equilibrium σ by adapting the proof of the "if" part of Proposition 7. Because $(b(\varsigma), s(\varsigma)) = (\bar{v}_b(\varsigma), \underline{v}_s(\varsigma))$ (see the proof of Proposition 3), $(\bar{v}_b(\varsigma), \underline{v}_s(\varsigma))$ is part of (17) and need to be determined simultaneously with $(y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma))$,[19] and in particular $A(\varsigma)$ satisfies

$$\begin{aligned} v/\delta \;=\; & \sum_{n=-N+1}^{2} 0.5\pi_n \{\rho[\alpha(\delta)u(y(v, v_n)) + \underline{v}_b(v, v_n)] + (1-\rho)\bar{v}_b(v, v_n) \\ & + \rho[\alpha(\delta)(1 - y(v_n, v)) + \bar{v}_s(v_n, v)] + (1-\rho)\underline{v}_s(v_n, v)\}. \end{aligned} \tag{36}$$

As the distribution of continuation values is non degenerate, we also need to modify $h_{k,t}$. Details are in the appendix. To complete the proof, let $v(\rho) = \sum \pi_n v_n$. Because $\lim_{\rho \to 1} \pi_1(\rho) = 1$, $\lim_{\rho \to 1} v(\rho) = 0.5[u(q^*) + 1 - q^*]$. ∎

In the context of supporting the first best (i.e., each seller transfers q^* when he is endowed) when agents are sufficiently patient, Propositions 7, 8, and 9 together deliver the following point. While renegotiation alone need not cause a problem, increasing the message space to accommodate another dimension of pairwise public information (represented by the seller's endowment realization in the present model) may not be effective and the degree of publicity of the other dimension (measured by ρ in the present

[19] This is not the case in the proofs of Propositions 6 and 7. In the first proof, we can set $(b(\varsigma), s(\varsigma)) = (\underline{v}_b(\varsigma), \bar{v}_s(\varsigma))$. In the second proof, $(b(\varsigma), s(\varsigma))$ is irrelevant.

model) may matter. This point is lost if, instead, each agent's type at each date is only pairwise public information—we lose Propositions 7 and 9 even if the agent's type is included in public messages.[20]

To complete this part, we note that for general ρ and δ if $u'(0)$ is sufficiently large then a constant positive transfer in some meetings can be assured. This can be shown by constructing an equilibrium in which the distribution of the continuation values has a two-point support $\{\underline{v}, \bar{v}\}$ ($\underline{v} < \bar{v}$)with equal mass.[21] The basic idea is as follows. When two agents meet at t and their start-of-t continuation values are in the support, there is a transfer $y \in (0, q^*]$ if and only if the buyer's start-of-t value is $\bar{v}$, the seller's is $\underline{v}$, and the seller is endowed. For two such agents, each agent's start-of-t+1 value is the same as his own start-of-t value if there is no transfer, and is the same as his meeting partner's start-of-t value if y is the transfer. So $\underline{v}/\delta = 0.25\rho\alpha(\delta) + (1 - 0.25\rho)\underline{v} + 0.25\rho[\alpha(\delta)(1 - y) + \bar{v}]$ and $\bar{v}/\delta = 0.5\rho\alpha(\delta) + (1 - 0.25\rho)\bar{v} + 0.25\rho[\alpha(\delta)u(y) + \underline{v}]$. To formally present such an equilibrium, the treatment is analogous to the one in the proof of Proposition 9; we omit the details.

6 Message trading and monetary exchange

Here we relate message trading to monetary exchange by replacing reports with money, a durable, divisible, and intrinsically useless object with a fixed stock, in the basic model. Specifically, in each pairwise meeting stage 1 is unchanged. At stage 2, a report r in a trading outcome is replaced with a feasible transfer of money l (from the buyer to the seller if $l \geq 0$), a report in an autarky outcome is replaced with free disposal of money, and no input into the reporting device in a non-terminal play is replaced with no transfer of money.

Following the literature on monetary matching models, our interest is in *the model with pairwise observed money*. That is, in a meeting each agent's money holdings and the transfer of money are pairwise public information, and the start-of-date-1 distribution of money is public information. Here to prescribe the stage-2 game form, $T_{k,t}$ depends on $(\theta_i^t, \phi_i^t)_{i \in I}$

[20] Proposition 4 (static characterization for RP equilibrium) remains valid.

[21] In many monetary matching models, agents are assumed to hold either 1 or 0 units of money, which makes the distribution of money holdings tractable. Regardless of the apparent similarity, the two-point support here is not due to any assumption that resembles the exogenous physical constraint on money holdings.

and $(m_{k,t}, m_{j,t}, \omega_{k,t}, c_{1,k,t})$ $(j = \phi_{k,t})$, where $m_{i,t}$ is agent i' money holdings at the start of his date-t meeting; and the strategy $\sigma_{k,t}$ conditions on $(\theta_i^t, \phi_i^t)_{i \in I}$, $(m_{k,t}, m_{j,t}, \omega_{k,t})$, and actions already taken by k and j during the meeting. The equilibrium concepts are then adapted from Definitions 3, 5, and 6; here at any information set k has the belief that the measure of agents with off-path holdings is zero. Following the literature, we restrict attention to equilibria in which money is never disposed.

Proposition 10 *If f is supported by an equilibrium in the model with pairwise observed money, then it is supported by an equilibrium in the basic model.*

Proof. For convenience of exposition, we first introduce *the model with publicly observed money*: Each agent's money holdings and the transfer of money in the meeting are public information; $T_{k,t}$ depends on $(\theta_i^t, \phi_i^t, m_i^t)_{i \in I}$ and $(\omega_{k,t}, c_{1,k,t})$, where $m_i^t = (i, m_{i,1}, ...m_{i,t})$; and $\sigma_{k,t}$ conditions on $(\theta_i^t, \phi_i^t, m_i^t)_{i \in I}$, $\omega_{k,t}$, and actions already taken by k and $\phi_{k,t}$ during the meeting.

Now it suffices to establish (a) If f is supported by an equilibrium σ for some T in the model with publicly observed money, then f is supported by an equilibrium σ' for some T' in the basic model; and (b) If f is supported by an equilibrium σ for some T in the model with pairwise observed money, then f is supported by an equilibrium σ' for some T' in the model with publicly observed money.

For (a), suppose the hypothesis holds. Let $\lambda : \mathbb{R} \to (0, 1)$ be a strictly increasing function. Let $\vartheta : [0, 1] \to \mathbb{R}$ be defined by $\vartheta(x) = \lambda^{-1}(x)$ if $x \in (0, 1)$ and $\vartheta(1) = \vartheta(0) = 0$. To describe T' and σ', fix $\gamma_t \equiv (\theta_i^t, \phi_i^t, r_i^{t-1})_{i \in I}$, k, and $\omega_{k,t}$. Given γ_t, let $m_{i,\tau+1} = m_{i,\tau} + (2\theta_{i,\tau} - 1) r_{i,s,\tau}^1 \vartheta(r_{i,b,\tau}^2)$ if $1 \le \tau < t$, where $m_{i,1}$ is i's initial money holdings in the model with publicly observed money; let $\gamma_t^m = (\theta_i^t, \phi_i^t, m_i^t)_{i \in I}$. In the game form prescribed by $T'_{k,t}$ (following $c_{1,k,t}$), k and $j = \phi_{k,t}$ simultaneously propose an outcome. If k and j propose the same feasible (y, r), then (y, r) is the outcome; otherwise, autarky is reached. Actions specified by $\sigma'_{k,t}$ are as follows. At stage 1 take the same action as specified by $\sigma_{k,t}$ given $(\gamma_t^m, \omega_{k,t})$. Following $c_{1,k,t}$, at stage 2 propose $(\bar{y}, \bar{r})$ with $\bar{r} = (\omega_{k,t}, \lambda(\bar{l}), \omega_{k,t}, \lambda(\bar{l}))$, where $(\bar{y}, \bar{l})$ is the outcome specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ in the game form prescribed by $T_{k,t}$ given $(\gamma_t^m, \omega_{k,t}, c_{1,k,t})$. In autarky announce $(0, 0)$.

To see that σ' is an equilibrium, it suffices to show that given $(\gamma_t, \omega_{k,t}, c_{1,k,t})$, by taking the value of the l transfer of money in σ given γ_t^m as the value of r with $r_s^1 \vartheta(r_b^2) = l$, k does not gain by proposing a different outcome at stage

2 if all other agents do not deviate from t on and there is no further deviation of k. But if k can gain, he can gain by deviating to autarky in σ given $(\gamma_t^m, \omega_{k,t}, c_{1,k,t})$. As σ' is an equilibrium, given $(\theta_i^t, \phi_i^t, \omega_i^t)_{i\in I}$, on-path plays of σ' and σ generate the same transfers of goods (notice that if on-path plays of σ generate $(m_i^{t+1})_{i\in I}$, then on-path plays of σ' generate $(r_i^t)_{i\in I}$ such that $m_{i,\tau+1} = m_{i,\tau} + (2\theta_{i,\tau} - 1)r_{i,s,\tau}^1 \vartheta(r_{i,b,\tau}^2)$, $1 \leq \tau \leq t$).

For (b), suppose the hypothesis holds. Let i's initial holdings in both models be the same $m_{i,1}$. To describe T' and σ', fix $\gamma_t^m \equiv (\theta_i^t, \phi_i^t, m_i^t)_{i\in I}$, k, and $\omega_{k,t}$. In the game form prescribed by $T'_{k,t}$ (following $c_{1,k,t}$), k and $j = \phi_{k,t}$ simultaneously propose an outcome. If k and j propose the same feasible (y, l), then (y, l) is the outcome; otherwise, autarky is reached. Actions specified by $\sigma'_{k,t}$ are as follows. At stage 1 take the same action as specified by $\sigma_{k,t}$ given $\{(\theta_i^t, \phi_i^t)_{i\in I}, m_{k,t}, m_{j,t}, \omega_{k,t}\}$. Following $c_{1,k,t}$, at stage 2 propose $(\bar{y}, \bar{l})$ if $\#\{i \in I : m_{i,t} \neq \hat{m}_{i,t}\} \leq 2(t-1)$, where $(\bar{y}, \bar{l})$ is the outcome specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ in the game form prescribed by $T_{k,t}$ given $\{(\theta_i^t, \phi_i^t)_{i\in I}, m_{k,t}, m_{j,t}, \omega_{k,t}, c_{1,k,t}\}$, and $\hat{m}_{i,t}$ is some holdings of i at the start of t generated by on-path plays of σ given (θ_i^t, ϕ_i^t); propose $(0, 0)$ otherwise. In autarky keep $m_{k,t}$. The remaining argument is analogous to the counterpart for (a). ■

The logic behind Proposition 10 is simple. Reports support a larger message space than money. So some component of the information content z' on an information set provided by reports to which a strategy profile σ' responds in the basic model can be identified as the information content z provided by money to which a strategy profile σ responds in the model with pairwise observed money (and any z can be identified as a component of some z'). When we let the response of σ' to z' be the response of an equilibrium σ to z, we obtain an equilibrium σ'. This logic does not rely on σ being only an equilibrium, but not a CP or RP equilibrium, so the proposition holds if we replace equilibrium with CP or RP equilibrium.

Now we provide two remarks on three related results, Proposition 10, the "only-if" part of Proposition 1, and the money-is-memory result in Kocherlakota [20]. First, if we state "Money is memory" as "Money is essential only if memory, the information about one's past actions that is accessible to his direct and indirect matching partners, is absent," then the "only-if" part of Proposition 1 and Proposition 10, respectively, can be stated as "Public messages are essential only if perfect monitoring (or, equivalently, the information about one's past actions that is accessible to the public) is absent" and "Money is essential only if public messages are absent."

Second, while our results imply "Money is essential only if perfect monitoring is absent," there is a counterexample to this conclusion in [20]. While the class of models in [20] is more general than the model here, the basic logic for Propositions 1 and 10 suggests that our conclusion fails in the example of [20] because there an agent's knowledge of the physical environment (the matching history in specific) is different under money than under perfect monitoring. Our conclusion seems to hold if such asymmetry in knowledge is eliminated.

Given Proposition 10, it is natural to ask whether reports can support more allocations than money. The next result serves the illustrative purpose.

Proposition 11 *Let (w, q, f) be given by Proposition 5. If $\rho < 1$, then f is not supported by any equilibrium in the model with pairwise observed money.*

Proof. See the appendix. ■

7 Concluding remarks

In our study we let the creation of messages be part of the risk-sharing stage game. When the stage game precedes the creation of messages, we can adapt our non-sequential model to a sequential model as follows. The transfer and consumption of the endowed good occur at stage 1; messages are input into the reporting device at stage 2; and at stage 2 there is another valuable object (which does not exist in our non-sequential model), say transferable utility, available for message trading. In the sequential model, the counterpart of Proposition 6 for CP equilibrium should hold, because in a meeting each agent at stage 2 can condition his action on the stage-1 transfer. But with renegotiation the set of stage-2 trades does not depend on the stage-1 transfer and, hence, the stage-1 transfer must be zero in equilibrium. To depart from this extreme result in the sequential model, the set of stage-2 trades should depend on the stage-1 transfer and, with such dependence, counterparts of propositions in subsections 5.2-5.3 should hold if messages are still essential. To obtain such dependence, one may consider employing hard evidence (which does not exist in our non-sequential model). Specifically, if a seller makes a transfer at stage 1, then he processes hard evidence that can confirm the stage-1 transfer. But how can messages be essential when hard evidence is available? A plausible answer is that examining evidence is costly. This seems interesting enough to be pursued in a separate study.

Two sorts of goods-message trading models, sequential and non-sequential, may be integrated to study non-cash payment methods such as checks and credit cards. While checks and cards may reveal the transfer of goods to outsiders as does currency, they do differ from currency. Specifically, for checks and cards to be used, peoples' payment histories ought to be accessible to outsiders for some finite cost. Otherwise, in the case of checks, the buyer can default on the payment by, say, depleting his checking account before the check is cleared. But if the buyer's default cannot be known to outsiders, the seller will not accept the buyer's check in the first place. Hence, for checks and cards there are two tiers of information revelation: a primary tier pertaining to transfers of goods and a secondary tier pertaining to payment histories. (For currency the secondary tier is irrelevant.) Notably, the secondary-tier information revelation may also be subject to message trading. For instance, in the case of checks, after the buyer pays for the seller's good with a check (the primary-tier information revelation), the buyer takes actions that affect the check clearing; then the buyer and the seller (or a third party) take actions that affect messages about the check clearing.[22] Hence, the two-tier revelation would naturally be represented by some integration of the sequential and non-sequential models.

Finally, our study suggests some features for models of coexisting payment methods and the evolution of their coexistence. Specifically, non-cash payment methods do not eliminate the informational friction that gives currency a role; non-cash payments would be substituted for cash payments when the substitution is socially beneficial; and the benefits would in part be driven by technological and institutional innovation that reduces the costs of sharing non-cash payment histories. In such a model, coexisting payment methods evolve to a cashless limit as goods are not paid by currency at all; the model with publicly observed money in the last section is such a limit.[23] While different cashless limits would be obtained in models with the suggested features, they are all conceptually different from models in which

[22] For example, the buyer pays some currency to the seller for not reporting a default.

[23] In that model each payment is public information so money cannot be interpreted as currency in the usual sense (while it may in the model with pairwise observed money). Public payment history seems to be a natural consequence of the ideal banking system described by Wicksell [37].

transaction frictions that are overcome by currency disappear.[24]

Appendix

The proof of Proposition 1

For the "only-if" part, let f be supported by an equilibrium σ for some T in the basic model. To describe σ' in the benchmark, first fix $\mu_t \equiv (\theta_i^t, \phi_i^t, \omega_i^{t-1})_{i\in I}$ and $(q_i^{t-1})_{i\in I}$. Given μ_t, let $(r_i^{t-1})_{i\in I}$ be the on-path report history in the basic model generated by σ; let $\gamma_t = (\theta_i^t, \phi_i^t, r_i^{t-1})_{i\in I}$. Given μ_t and $(q_i^{t-1})_{i\in I}$, for $\tau \in \{1, ..., \max\{1, t-1\}\}$ define by induction $z_{i,\tau} \in \{0,1\}$ so that $Z_\tau^0 = \{i \in I : z_{i,\tau} = 0\}$ has at most 1 element; we set $z_{i,0} = 1$, all i, and let $\#Z_0^0 = 0$. When $\#Z_{\tau-1}^0 = 1$, let $(\hat{r}_i^{\tau-1})_{i\in I}$ be a report history in the basic model in which the only deviation from σ up to the start of τ, under μ_τ, is made by the agent in $Z_{\tau-1}^0$ at $\tau - 1$. Given $\hat{\gamma}_\tau \equiv (\theta_i^\tau, \phi_i^\tau, \hat{r}_i^{\tau-1})_{i\in I}$ and $\omega_{k,\tau}$, let $\hat{y}_{k,\tau}$ be the transfer specified by $\sigma_{k,\tau}$ and $\sigma_{j,\tau}$ ($j = \phi_{k,\tau}$). Then set $d_{k,\tau} = 0$ iff either $z_{k,\tau-1} \cdot z_{j,\tau-1} \cdot \theta_{k,\tau} = 1$ and $q_{k,\tau} < \hat{y}_{k,\tau}$, or $k \in Z_{\tau-1}^0$. When $\#Z_{\tau-1}^0 = 0$, set $d_{k,\tau} = 0$ iff $\theta_{k,\tau} = 1$ and $q_{k,\tau} < y_{k,\tau} \equiv f_{k,\tau}(\mu_\tau, \omega_{k,\tau})$. Letting $D_\tau = \{k \in I : d_{k,\tau} = 0\}$, if $\#D_\tau \leq 1$, then $z_{i,\tau} = d_{i,\tau}$; if $D_\tau = \{k_1, k_2\}$ with $z_{k_1,\tau-1} = 0$ and $z_{k_2,\tau-1} = 1$, then $z_{i,\tau} = 0$ iff $i = k_2$; for all other D_τ, $z_{i,\tau} = 1$. Next fix $(k, \omega_{k,t})$. Actions specified by $\sigma'_{k,t}$ when $\theta_{k,t} = 1$ are as follows. At stage 1 choose $c_{1,k,t} = 0$. At stage 2, if $\#Z_{t-1}^0 = 0$ and $1 - c_{1,k,t} \leq y_{k,t}$, transfer $y_{k,t}$; if $\#Z_{t-1}^0 \cdot z_{k,t-1} \cdot z_{j,t-1} = 1$ and $1 - c_{1,k,t} \leq \hat{y}_{k,t}$, transfer $\hat{y}_{k,t}$; otherwise, transfer 0. To see that σ' is an equilibrium, the case worth of checking is that given $(\mu_t, \omega_{k,t}, c_{1,k,t})$ and $(q_i^{t-1})_{i\in I}$ with $z_{k,\tau-1} \cdot z_{j,\tau-1} \cdot \theta_{k,\tau} = 1$ and $y_{k,t} \geq 1 - c_{1,k,t} > 0$ if $\#Z_{t-1}^0 = 0$, or $\hat{y}_{k,t} \geq 1 - c_{1,k,t} > 0$ if $\#Z_{t-1}^0 = 1$, k does not gain by deviating at stage 2 if all other agents do not deviate from t on and there is no further deviation of k. Transferring $y_{k,t}$ when $\#Z_{t-1}^0 = 0$ ($\hat{y}_{k,t}$ when $\#Z_{t-1}^0 = 1$, respectively) in σ' gives k the same payoff as transferring $y_{k,t}$ given $(\gamma_t, \omega_{k,t})$ ($\hat{y}_{k,t}$ given $(\hat{\gamma}_t, \omega_{k,t})$, respectively) in σ. His payoffs from a smaller transfer y in σ' and in σ, respectively, are $(1-\delta)\cdot(1-y) + \delta \cdot 0.5\rho$ and $(1-\delta)\cdot(1-y) + \delta \cdot v$ for some $v \geq 0.5\rho$. Because σ is an equilibrium, k does not gain by deviating from $\sigma'_{k,t}$.

[24]For example, the cashless limits in Calvacanti and Wallace [5] and Kocherlakota and Wallace [22] are matching models with perfect monitoring. Also see Woodford [38] in footnote 7.

The completion of the proof of Proposition 4

Here we prove the six intermediate results in the main text.

Result 1. *In (x, r_x), $x = 1 - c$; there exists a report $\breve{r}_x$ such that $(0, \breve{r}_x)$ is Pareto dominated by (x, r_x), and each of k and j can obtain $(0, \breve{r}_x)$ given the other does not deviate.*

Let $v_i = v_i(r_x)$ and $v_i^* = v_i(r^*)$. We first show $x = 1 - c$. By Proposition 2, $\Delta_j = v_j^* - v_j > 0$ and $\Delta_k = v_k - v_k^* > 0$. By efficiency of (y^*, r^*) in the game form following the on-path $c_{1,k,t}$, $(y^*, 0) \in \arg\max[\alpha(\delta)u(q) + z\Delta_k]$ for $(q, z) \in [0, y^*] \times [0, 1]$ and $z\Delta_j = \alpha(\delta)(y^* - q)$. If $x < 1 - c$, then by efficiency of (x, r_x) following $c_{1,k,t} = c$, $(x, 0) \in \arg\max[\alpha(\delta)u(q) - z\Delta_k]$ for $(q, z) \in [x, 1-c] \times [0, 1]$ and $z\Delta_j = \alpha(\delta)(q - x)$. Hence for z in a neighborhood of 0, $0 \in \arg\max_z \alpha(\delta)u[y^* - z\Delta_j/\alpha(\delta)] + z\Delta_k$ so $u'(y^*)\Delta_j \geq \Delta_k$, and also $0 \in \arg\max_z \alpha(\delta)u[x + z\Delta_j/\alpha(\delta)] - z\Delta_k$ so $\Delta_k \geq u'(x)\Delta_j$. But $u'(y^*) \geq u'(x)$ contradicts to $y^* > x$ and $u'' < 0$. So $x = 1 - c$.

Next let $\tilde{r}_x = (\tilde{r}_{x,b}, \tilde{r}_{x,s})$ be the report specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ following the history that k adheres to $\sigma_{k,t}$ but j makes some play(s) leading to autarky; and let $\hat{r}_x = (\hat{r}_{x,b}, \hat{r}_{x,s})$ be the report specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ following the history that j adheres to $\sigma_{j,t}$ but k makes some play(s) leading to autarky. Without loss of generality assume $v_j(\tilde{r}_x) \leq v_j(\hat{r}_x)$. Let $\breve{r}_x = (\tilde{r}_{x,b}, \hat{r}_{x,s})$ so each of k and j can obtain $(0, \breve{r}_x)$ given the other does not deviate. Let $\tilde{v}_i = v_i(\tilde{r}_x)$, $\hat{v}_i = v_i(\hat{r}_x)$, $\breve{v}_i = v_i(\breve{r}_x)$, $l_k = \breve{v}_k - v_k$, and $l_j = v_j - \breve{v}_j$. For σ to be an equilibrium, $v_j - \tilde{v}_j \geq \alpha(\delta)x$, $\hat{v}_k - v_k \leq \alpha(\delta)u(x)$, $\breve{v}_j \leq \tilde{v}_j$, and $\breve{v}_k \leq \hat{v}_k$ (see the similar argument in the proof of Proposition 4). By efficiency of (x, r_x), $(x, 0) \in \arg\max[\alpha(\delta)u(q) + zl_k]$ for $(q, z) \in [0, x] \times [0, 1]$ and $zl_j = \alpha(\delta)(x - q)$. Hence for z in a neighborhood of 0, $0 \in \arg\max_z \alpha(\delta)u[x - zl_j/\alpha(\delta)] + zl_k$ so $u'(x)l_j \geq l_k$. By $u'' < 0$, this rules out $(\alpha(\delta)u(x), \alpha(\delta)x) = (l_k, l_j)$. So $(0, \breve{r}_x)$ is Pareto dominated by (x, r_x).

Result 2. *$-v(.)$ and $w(.)$ are nondecreasing and continuous on $(0, y^*]$; $v(0) - v(y^*) < \alpha(\delta)u(y^*)$ and $w(y^*) - w(0) \geq \alpha(\delta)y^*$ with $(v(0), w(0)) = \lim_{x \downarrow 0}(v(x), w(x))$.*

Fix $0 < x \leq y^*$, let y be sufficiently close to x, and refer to as Games X and Y for the game forms following $c = 1 - x$ and $c = 1 - y$, respectively. Let $V_p = \alpha(\delta)u(p) + v(p)$ and $W_p = w(p) - \alpha(\delta)p$ for $p \in \{x, y\}$.

First consider monotonicity and let $y < x$. Because (x, r_x) and (y, r_y) are efficient in relevant games, we rule out (i) $[v(y) - v(x)][w(y) - w(x)] > 0$, and (ii) $[v(y) - v(x)][w(y) - w(x)] = 0$ and $(v(y), w(y)) \neq (v(x), w(x))$. Now suppose by contradiction that either $-v(.)$ or $w(.)$ is strictly decreasing and so it must be (M0) $v(y) < v(x)$ and $w(y) > w(x)$.

In Game X, let $\tilde{r} = (\tilde{r}_b, \tilde{r}_s)$ be the report specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ following the history that k adheres to $\sigma_{k,t}$ but j makes some play(s) leading to autarky. In Game Y, let $\hat{r} = (\hat{r}_b, \hat{r}_s)$ be the report specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ following the history that j adheres to $\sigma_{j,t}$ but k makes some play(s) leading to autarky. Let $\breve{r} = (\tilde{r}_b, \hat{r}_s)$. For j not to deviate in game X, $W_x \geq \breve{v}_j$, implying $(0, \breve{r})$ not efficient in game X. For, otherwise, $V_x \leq \breve{v}_k$, which together with (M0) and $y < x$ implies $V_y < \breve{v}_k$, i.e., k has a beneficial deviation in game Y. By the same argument (for k not to deviate and $(0, \breve{r})$ to be efficient in game Y, j has a beneficial in game X), $(0, \breve{r})$ is not efficient in game Y. For $p \in \{x, y\}$ denote by (p', r'_p) the Nash solution with the disagreement point $(0, \breve{r})$ in Game P$\in$ {Y,X}. Let $V'_p = \alpha(\delta)u(p') + v_k(r'_p)$, $W'_p = v_j(r'_p) - \alpha(\delta)p'$, $\breve{v}_i = v_i(\breve{v})$, $\breve{V}_p = V_p - \breve{v}_k$, $\breve{W}_p = W_p - \breve{v}_j$, $\breve{V}'_p = V'_p - \breve{v}_k$, and $\breve{W}'_p = W'_p - \breve{v}_j$. By definition, $\breve{V}'_x, \breve{W}'_x, \breve{V}'_y, \breve{W}'_y > 0$.

Now we have (M1) $\breve{V}_y \geq \breve{V}'_y$ and $\breve{W}_x \geq \breve{W}'_x$ (renegotiation does not make k/j better off in Game Y/X); (M2) $\breve{V}_p \breve{W}_p \leq \breve{V}'_p \breve{W}'_p$ for each p ((p', r'_p) is the Nash solution); and (M3) $u'(p')\breve{W}'_p \geq \breve{V}'_p$ and equal if $p' < p$ for each p (the first order condition on p'). By (M1) and (M2), $\breve{W}_y \leq \breve{W}'_y$ and $\breve{V}_x \leq \breve{V}'_x$. By (M0), $\breve{V}_x > \breve{V}_y$ and $\breve{W}_x < \breve{W}_y$. Putting these four inequalities and (M1) together, we have (M4) $\breve{V}'_x \geq \breve{V}_x > \breve{V}_y \geq \breve{V}'_y$ and $\breve{W}'_x \leq \breve{W}_x < \breve{W}_y \leq \breve{W}'_y$. By (M4), $\breve{V}'_x/\breve{W}'_x > \breve{V}'_y/\breve{W}'_y$. This and (M3) imply either $y' = y$ or $x' < y' < y$. But if $x' < y'$ then (y', r'_y) cannot be the Nash solution in Game Y. So $y' = y$ and then $x' \leq x$, (M4), and (M0) imply $v_k(r'_y) \leq v(y) < v(x) \leq v_k(r'_x)$ and $v_j(r'_y) \geq w(y) > w(x) \geq v_j(r'_x)$. Hence $\Delta_k = v_k(r'_x) - v_k(r'_y) > 0$ and $\Delta_j = v_j(r'_y) - v_j(r'_x) > 0$. But then $0 \in \arg\max_z(\breve{V}'_x - z\Delta_k)(\breve{W}'_x + z\Delta_j)$ and $0 \in \arg\max_z(\breve{V}'_y + z\Delta_k)(\breve{W}'_y - z\Delta_j)$ imply $\breve{V}'_x/\breve{W}'_x \leq \Delta_k/\Delta_j$ and $\breve{V}'_y/\breve{W}'_y \geq \Delta_k/\Delta_j$, a contradiction.

Next for continuity suppose by contradiction that either $v(.)$ or $w(.)$ is not continuous at x. By $G_j(0; r_y, x)[v(y) - v(x)] \leq G_k(0; r_y, x)[w(x) - w(y)]$ (see (3)) and monotonicity, $v(x) < \lim_{y\uparrow x} v(y) \Rightarrow w(x) > \lim_{y\uparrow x} w(y)$ and $w(x) < \lim_{y\downarrow x} w(y) \Rightarrow v(x) > \lim_{y\downarrow x} v(y)$. Using $G_j(0; r_x, y)[v(x) - v(y)] \leq G_k(0; r_x, y)[w(y) - w(x)]$, we have the opposite implications. Without loss of generality we assume (C0) $v(x) < \lim_{y\uparrow x} v(y)$ and $w(x) > \lim_{y\uparrow x} w(y)$ ($v(x) > \lim_{y\downarrow x} v(y)$ and $w(x) < \lim_{y\downarrow x} w(y)$ can be dealt with analogously).

In Game X, let $\bar{r} = (\bar{r}_b, \bar{r}_s)$ be the report specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ following the history that j adheres to $\sigma_{j,t}$ but k makes some play(s) to leading to autarky. In Game Y with $y < x$, let $\check{r} = (\check{r}_b, \check{r}_s)$ be the report specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ following the history that k adheres to $\sigma_{k,t}$ but j makes some play(s) to leading to autarky. Let $\check{r} = (\bar{r}_b, \check{r}_s)$. While this $\check{r}$ need not be the same as the $\check{r}$ in the above proof for monotonicity, let (p', r'_p), V'_p, W'_p, $\check{v}_i$, $\check{V}_p$, $\check{W}_p$, $\check{V}'_p$, and $\check{W}'_p$ be defined as in the above. Adapting the above argument, we can show that $(0, \check{r})$ is not efficient in either game (here we use $W_x \leq \check{v}_j \Rightarrow W_y < \check{v}_j$ and $V_y \leq \check{v}_k \Rightarrow V_x < \check{v}_k$ as y is close to x). Then again, $\check{V}'_x, \check{W}'_x, \check{V}'_y, \check{W}'_y > 0$.

Now we have (C1) $\check{V}_x \geq \check{V}'_x$ and $\check{W}_y \geq \check{W}'_y$, and have (C2) and (C3) the same as (M2) and (M3), respectively, in the proof for monotonicity. By (C1) and (C2), $\check{V}_y \leq \check{V}'_y$ and $\check{W}_x \leq \check{W}'_x$. By (C0) and y being close to x, $\check{V}_x < \check{V}_y$ and $\check{W}_x > \check{W}_y$. Putting these four inequalities and (C1) together, we have (C4) $\check{V}'_x \leq \check{V}_x < \check{V}_y \leq \check{V}'_y$ and $\check{W}'_x \geq \check{W}_x > \check{W}_y \geq \check{W}'_y$. By (C4), $\check{V}'_x/\check{W}'_x > \check{V}'_y/\check{W}'_y$. We claim $v(x) \geq v_k(r'_x)$. (This is the place where the proof for continuity differs from the proof for monotonicity.) Then efficiency of (x', r'_x) implies $w(x) \leq v_j(r'_x)$. So by (C0) and (C4), $v_k(r'_y) \geq v(y) > v(x) \geq v_k(r'_x)$ and $v_j(r'_y) \leq w(y) < w(x) \leq v_j(r'_x)$ ($\check{V}_y \leq \check{V}'_y \Rightarrow v_k(r'_y) \geq v(y)$ and $\check{W}_y \geq \check{W}'_y \Rightarrow v_j(r'_y) \leq w(y)$). Hence $d_k = v_k(r'_y) - v_k(r'_x) > 0$ and $d_j = v_j(r'_x) - v_j(r'_y) > 0$. But then $0 \in \arg\max_z(\check{V}'_x - zd_k)(\check{W}'_x + d_j)$ and $0 \in \arg\max_z(\check{V}'_y + zd_k)(\check{W}'_y - zd_j)$ imply $\check{V}'_x/\check{W}'_x \leq d_k/d_j$ and $\check{V}'_y/\check{W}'_y \geq d_k/d_j$, a contradiction.

For the claim, suppose the converse that $l_k = v_k(r'_x) - v(x) > 0$. Then efficiency of (x, r_x) implies $l_j = w(x) - v_j(r'_x) > 0$. Also by $l_k > 0$, (C1) implies $x' < x$ so (C3) implies $u'(x') = \check{V}'_x/\check{W}'_x$. With $\check{V}'_x/\check{W}'_x \leq l_k/l_j$ following from $0 \in \arg\max_z(\check{V}'_x - zl_k)(\check{W}'_x + zl_j)$, $u'(x') \leq l_k/l_j$. But efficiency of (x, r_x) implies $(x, 0) \in \arg\max[\alpha(\delta)u(q) + zl_k]$ for $(q, z) \in [0, x] \times [0, 1]$ and $zl_j = \alpha(\delta)(x - q)$ so $0 \in \arg\max_z \alpha(\delta)u[x - zl_j/\alpha(\delta)] + zl_k$ or $u'(x) \geq l_k/l_j$, a contradiction (above we have $x' < x$ and $u'(x') \leq l_k/l_j$).

With continuity, $w(y^*) - w(0) \geq \alpha(\delta)y^*$ follows from Proposition 2.

Finally we turn to $v(0) - v(y^*) < \alpha(\delta)u(y^*)$. Let $\mathring{r} = (\mathring{r}_b, \mathring{r}_s)$ be the report specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ in the game form following $c_{1,k,t} = 1$. For j not to deviate in this game form, $v_j(\mathring{r}) \geq v_j(r)$ with $r = (\mathring{r}_b, r_s)$ for any $r_s \in \{0, 1\} \times [0, 1]$ and, if $v_k(\mathring{r}) \geq v_k(r)$ and $[v_k(\mathring{r}) - v_k(r)] + [v_j(\mathring{r}) - v_j(r)] > 0$ then any $(0, r')$ that maximizes the Nash product with $(0, r)$ as the disagreement point and 0 units of goods remaining satisfies $v_j(\mathring{r}) \geq v_j(r')$ (j does not gain by renegotiation) and $v_k(r') \geq v_k(\mathring{r})$ (the renegotiated outcome is efficient).

We note that the above proof for monotonicity does not rely on $y > 0$. (The only use of $y > 0$ there is to make (M3) sensible, and the only use of (M3) is to obtain $y' = y$, which is trivial in case $y = 0$.) So $v_k(\mathring{r}) \geq v(0)$.

Now without loss of generality, suppose that the on-path $c_{1,k,t}$ is equal to $1 - y^*$ and consider the game form following the on-path $c_{1,k,t}$. In this game form, let r_s be the announcement specified by $\sigma_{j,t}$ following the history that j adheres to $\sigma_{j,t}$ but k makes some play(s) leading to autarky; let $r = (\mathring{r}_b, r_s)$. We have $v_k(r) \leq \alpha(\delta)u(y^*) + v(y^*)$ (k does not to deviate to autarky and announce $\mathring{r}_b$.) If $v_k(\mathring{r}) < v_k(r)$ then $v(0) < \alpha(\delta)u(y^*) + v(y^*)$ (recall that $v_k(\mathring{r}) \geq v(0)$). So suppose $v_k(\mathring{r}) \geq v_k(r)$. Notice that $v_j(r) \leq v_j(\mathring{r})$ (the definition of $\mathring{r}$) and $v_j(\mathring{r}) \leq -\alpha(\delta)y^* + w(y^*)$ (Proposition 2).

We proceed by assuming that $(0, r)$ is not efficient in the game form (treating $v_k(r) = \alpha(\delta)u(y^*) + v(y^*)$ and $v_j(r) = -\alpha(\delta)y^* + w(y^*)$ as a special case). Let (y, r'') maximize the Nash product with $(0, r)$ as the disagreement point and y^* units of goods remaining; notice that $\alpha(\delta)u(y^*) + v(y^*) \geq \alpha(\delta)u(y) + v_k(r'')$ (k does not gain by renegotiation) and $-\alpha(\delta)y^* + w(y^*) \leq -\alpha(\delta)y + v_j(r'')$ ((y, r'') is efficient). If $v_j(\mathring{r}) = v_j(r)$ and $v_k(\mathring{r}) = v_k(r)$, then by $\alpha(\delta)u(y) + v_k(r'') > v_k(r)$ ((y, r'') maximizes the Nash product), $v(0) < \alpha(\delta)u(y^*) + v(y^*)$ (recall that $v_k(\mathring{r}) \geq v(0)$). So suppose $(0, r)$ is Pareto dominated by $(0, \mathring{r})$ (as noted above $v_j(\mathring{r}) \geq v_j(r)$) and let $(0, r')$ maximize the Nash product with $(0, r)$ as the disagreement point and 0 units of goods remaining. We claim that $\alpha(\delta)u(y) > v_k(r') - v_k(r'') \equiv d_k$. Then $v(0) < \alpha(\delta)u(y^*) + v(y^*)$ (recall that $v_k(r') \geq v_k(\mathring{r}) \geq v(0)$).

To see the claim, suppose the converse so $d_j \equiv v_j(r'') - v_j(r') \geq \alpha(\delta)y$ ((y, r'') is efficient). If $d_j > 0$, then by definitions of (y, r'') and $(0, r')$, $0 \in \arg\max_z[\alpha(\delta)u(y) + v_k(r'') + zd_k - v_k(r)][-\alpha(\delta)y + v_j(r'') - zd_j - v_j(r)]$ and $0 \in \arg\max_z[v_k(r') - zd_k - v_k(r)][v_j(r') + zd_j - v_j(r)]$. It follows that

$$\frac{\alpha(\delta)u(y) + v_k(r'') - v_k(r)}{-\alpha(\delta)y + v_j(r'') - v_j(r)} \geq \frac{d_k}{d_j} \geq \frac{v_k(r') - v_k(r)}{v_j(r') - v_j(r)}.$$

So $\alpha(\delta)u(y) = d_k$ and $\alpha(\delta)y = d_j$. By $y > 0$ and efficiency of (y, r''), $0 \in \arg\max_z \alpha(\delta)u[y - zd_j/\alpha(\delta)] + zd_k$ so $u'(y) \geq d_k/d_j$, contradicting to $u'' < 0$ and $u(y)/y = d_k/d_j$. The same contradition follows if $d_k > 0$. If $d_k = d_j = 0$, then $y = 0$. By $v_j(r') \leq v_j(\mathring{r}) \leq -\alpha(\delta)y^* + w(y^*) \leq -\alpha(\delta)y + v_j(r'')$, $\Delta_j \equiv w(y^*) - v_j(r'') = \alpha(\delta)y^*$. It follows that $\Delta_k \equiv v_k(r'') - v(r^*) = \alpha(\delta)u(y^*)$ (recall that (y, r'') is efficient and $\Delta_k \leq \alpha(\delta)u(y^*)$). By efficiency of (y^*, r^*), $0 \in \arg\max_z \alpha(\delta)u[y^* - z\Delta_j/\alpha(\delta)] + z\Delta_k$ so $u'(y^*) \geq \Delta_k/\Delta_j$, contradicting to $u'' < 0$ and $u(y^*)/y^* = \Delta_k/\Delta_j$. This completes the proof.

Result 3. *(10)⇒ (11) and (12), and (11)⇔(13).*
By $dK(\tau;x)/d\tau = -h(\tau)K(\tau;x)$ and integration by parts, (10) yields

$$\begin{aligned} \hat{v}_n(x) &= m_n(x) - \int_0^x m_n(\tau)dK(\tau;x) \\ &= m_n(x) - m_n(\tau)K(\tau;x)|_0^x + \int_0^x K(\tau;x)m_n'(\tau)d\tau. \end{aligned} \quad (37)$$

From (37) (the second equality and $m_n(x) = m_n(\tau)K(\tau;x)|_0^x$), we obtain (11) and (12). From (37) and applying the same argument used in (37), we have $\hat{v}_1(x) = -\alpha(\delta)u(\tau)K(\tau;x)|_0^x + \alpha(\delta)\int_0^x K(\tau;x)u'(\tau)d\tau$ and further

$$\hat{v}_1(x) = -\alpha(\delta)u(x) + \alpha(\delta)\int_0^x K(\tau;x)u'(\tau)d\tau. \quad (38)$$

By $K(\tau;x) = H(\tau;x)\exp\{-\int_\tau^x \alpha(\delta)[\hat{w}(\zeta) - \alpha(\delta)\zeta]^{-1}d\zeta\}$, where

$$\begin{aligned} H(\tau;x) &= \exp\{-\int_\tau^x [\hat{w}'(\zeta) - \alpha(\delta)][\hat{w}(\zeta) - \alpha(\delta)\zeta]^{-1}d\zeta\} \\ &= \exp\{\int_x^\tau d\ln[\hat{w}(\zeta) - \alpha(\delta)\zeta]\} = [\hat{w}(\tau) - \alpha(\delta)\tau]/[\hat{w}(x) - \alpha(\delta)x], \end{aligned}$$

(38) yields (13).

Result 5. *If* $\hat{w}(0) = 0$ *then* $\hat{v}(0) - \hat{v}(y^*) > -\varphi(y^*, y^*, l_j^*, \delta)$.

Let $\hat{w}(0) = 0$. Then by (5) and $w(y^*) = v_j(r^*)$, $\hat{w}(y^*) - \hat{w}(0) = l_j^*$. So we can find $\bar{x} > 0$ with $\hat{w}(y^*) - \hat{w}(\bar{x}) \geq \alpha(\delta)y^*$. By $\hat{w}(0) = \lim_{x\downarrow 0}\hat{w}(x) = 0$ and $\hat{w}(x) - \alpha(\delta)x > 0$ for $x > 0$, $\hat{w}(.)$ is not constant over $(0,\epsilon]$ if $\epsilon > 0$. We claim that $l_k^* = \hat{v}(0) - \hat{v}(y^*)$. By the claim, $\hat{v}(.)$ is not constant over $(0,\epsilon]$ if $\epsilon > 0$. For, otherwise, by (4) and $v(y^*) = v_k(r^*)$ and $l_k^* = v(0) - v(y^*)$, $\hat{v}(x) = v(x) - v(0)$ is constant and so $\hat{v}(x) = 0$ for x around 0; but because $\hat{w}(.)$ is not constant around 0, (6) cannot hold for some x and x'. So for $\epsilon > 0$ there exists $\underline{x}(\epsilon) \in (0,\bar{x})$ with $\hat{v}(0) - \hat{v}(\underline{x}(\epsilon)) \in (0, \epsilon\hat{w}(\bar{x}))$.

Now let $\hat{v}_1(x) = \hat{v}(\underline{x}(\epsilon))$ for $x \leq \underline{x}(\epsilon)$ and $\hat{v}_1(x) = \hat{v}(x)$ for $x > \underline{x}(\epsilon)$. Also, let $\hat{w}_1(x) = \hat{w}(\bar{x})$ for $x \leq \bar{x}$ and $\hat{w}_1(x) = \hat{w}(x)$ for $x > \bar{x}$. By construction, $\hat{w}_1(y^*) - \hat{w}_1(0) \geq \alpha(\delta)y^*$, $\hat{v}_1(.)$ and $\hat{w}_1(.)$ are monotonic and continuous, and (6) holds if $(\hat{v}(.), \hat{w}(.))$ is replaced with $(\hat{v}_1(.), \hat{w}_1(.))$ as $\epsilon \to 0$. But by construction, $\hat{v}_1(0) - \hat{v}_1(y^*) < \hat{v}(0) - \hat{v}(y^*)$ and so the result follows from result 4.

For the claim, suppose $d_k = l_k^* - [\hat{v}(0) - \hat{v}(y^*)] > 0$. Fix $x' > 0$ so that $v(0) - v(x') < 0.5d_k$. Then as $x \to 0$, $[v(0) + d_k] + v(x') > 2[\alpha(\delta)u(x) + v(x)]$ and because $\hat{w}(.)$ is not constant around 0, $w(0) + w(x') > 2[w(x) - \alpha(\delta)x]$. But then (x, r_x) is not efficient in the game form following $c = 1 - x$ (j's continuation value is at least $w(0)$).

Result 6. *$\varphi_s(y, \delta)$ and $\varphi(x, y, l, \delta)$ have the properties stated in the proposition; and $\varphi_s(y, \delta)$ and $\varphi_b(y, l, \delta)$ are decreasing in δ.*

Let $\lambda(l) = [l/\alpha(\delta) - y]^{-1}$ and then from (2), we obtain

$$\varphi_1(x, y, l, \delta) \equiv \frac{\partial}{\partial x}\varphi(x, y, l, \delta) = -[\alpha(\delta)u(x) + \varphi(x, y, l, \delta)]\lambda(l), \quad (39)$$

$$\frac{\partial}{\partial x}\varphi_1(x, y, l, \delta) = -[\alpha(\delta)u'(x) + \varphi_1(x, y, l, \delta)]\lambda(l). \quad (40)$$

By (40), $x \mapsto \varphi(x, y, l, \delta)$ is concave iff $\alpha(\delta)u'(x) + \varphi_1(x, y, l, \delta) \geq 0$ or

$$l - \varphi(x, y, l, \delta)/u'(x) \geq \alpha(\delta)[y + u(x)/u'(x)] \ \forall x \in [0, y]. \quad (41)$$

Let $L = \{l : (41) \text{ holds}\}$. Because $\varphi(x, y, l, \delta) \leq 0$ (see (2)), (41) holds when $l \geq \alpha(\delta)\nu(y)$ so L is nonempty. Clearly L is closed so $\varphi_s(y, \delta)$ is defined. Letting $J(x, l) = u'(x) - \int_0^x \exp[-\lambda(l)(x-\tau)]\lambda(l)u'(\tau)d\tau$, then by (14), (41) $\Leftrightarrow$ $J(x, l) \geq 0$. When $u'(0)$ is finite, integration by parts implies that $J(x, l) \geq 0$ iff $u'(0) + \int_0^x u''(\tau)\exp[\lambda(l)\tau]d\tau \geq 0$; when $u'(0)$ is infinite, we have $J(x, l) \geq 0$ iff $\lim_{\epsilon \downarrow 0}\{u'(\epsilon) + \int_\epsilon^x u''(\tau)\exp[\lambda(l)\tau]d\tau\} \geq 0$. Either way, $l \in L$ if $l > \varphi_s(y, \delta)$, and $\varphi_s(y, \delta)$ is decreasing in δ. Also, by (14), $\varphi_b(y, l, \delta)$ is decreasing in δ. Next let $\underline{l} = \varphi_s(y, \delta)$ and $l_b = \varphi_b(y, l, \delta)$. Then (41), $-\varphi_1(y, y, \underline{l}, \delta)y > l_b$ (strict concavity of $\varphi(.)$), and $\underline{l} > \alpha(\delta)y$ imply $u'(y)[\underline{l} - \alpha(\delta)y] \geq [\alpha(\delta)u(y) - l_b]$ and $[\alpha(\delta)u(y) - l_b]\underline{l} > [\underline{l} - \alpha(\delta)y]l_b$. It follows that $\alpha(\delta)u(y)\underline{l}[2\underline{l} - \alpha(\delta)y]^{-1} > \alpha(\delta)\nu(y) - u'(y)\underline{l}$. Then some algebra yields $\underline{l} > 0.5\alpha(\delta)\nu(y)$. By (41) and continuity, $\underline{l} < \alpha(\delta)\nu(y)$.

The proof of Lemma 2

For part (i), we claim that $\bar{v}(\delta)$ is nondecreasing and right continuous in δ. Then the result follows from the claim, that $-\varphi_b(q, w - \underline{w}, \delta)$ and $\varphi_s(y, \delta)$ are continuous and decreasing in δ (see result 6 in the proof of Proposition 4), and $\lim_{\delta \to 1}\varphi_b(q, w - \underline{w}, \delta) = \lim_{\delta \to 1}\varphi_s(q, \delta) = 0$. For the claim, because $\kappa(y, \eta, \delta)$ is increasing in η and $\varphi_b(y, l_s, \delta)$ is decreasing in l_s, when $\bar{v}(\delta)$ is attained $\kappa(y, \eta, \delta) - \eta = \varphi_b(y, l_s, \delta)$ (i.e., the last two inequalities in (21) hold as equalities) and $\kappa(y, \eta, \delta) - \underline{w} = l_s$. Now fix δ_0 and let $\delta \geq \delta_0$. Let $\bar{v}(\delta_0)$ be attained by some $(y, \eta, l_b, l_s) \in \Gamma(\delta_0)$. Then let $\eta(\delta)$ and $l_s(\delta)$ satisfy $\kappa(y, \eta(\delta), \delta) = \bar{v}(\delta_0)$ and $l_s(\delta)/\alpha(\delta) = l_s/\alpha(\delta_0)$. Let $l_b(\delta) = \varphi_b(y, l_s(\delta), \delta)$. By (2), $\varphi_b(y, l_s(\delta), \delta)/\alpha(\delta)$ is constant in δ; by computation, $[\bar{v}(\delta_0) - \eta(\delta)]/\alpha(\delta)$ is constant in δ. Hence $\bar{v}(\delta_0) - \eta(\delta) = \varphi_b(y, l_s(\delta), \delta)$. This, $\bar{v}(\delta_0) - \underline{w} = l_s$, and $\alpha'(\delta) < 0$ imply $(y, \eta(\delta), l_b(\delta), l_s(\delta)) \in \Gamma(\delta)$ and, hence, monotonicity. Given monotonicity, right continuity of $\bar{v}(.)$ follows from that $\Gamma(.)$ is compact valued and upper hemicontinuous. For part (ii), if $\sup V \in V$ then $\sup V \leq \bar{v}(\delta)$ is clear; otherwise, fix $\{v_n\} \subset V$ with $\lim v_n = \sup V$ and passing to the limit yields $\sup V \leq \bar{v}(\delta)$.

Completion of the proof of Lemma 3

Refer to the proof of Lemma 1. In cases (a), (d), and (e), and in cases (b) and (c) when $v = w$, let $(y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma))$ be the same as in the previous proof. In case (b) when $v > w$ let $(y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma)) = (0, v, w)$. In case (c) with $v > w$, let $\bar{v}$ be attained by some $(y, \eta, l_b, l_s) \in \Gamma(\delta)$. We claim that there exist some $(x, \zeta) \in [0, y] \times [\underline{v}, \eta]$ such that $v = \kappa(x, \zeta, \delta)$. Then set $(y(\varsigma), \underline{v}_b(\varsigma), \bar{v}_s(\varsigma)) = (x, \zeta, \bar{v}(\delta))$, assuring (17). For the claim, first suppose $\eta \leq w$. Then by $(q, w, \bar{v} - w, w - \underline{v}) \in \Gamma(\delta)$ (Lemma 2 (i)), $y \geq q$ and so by continuity $\{\kappa(x, \eta, \delta) : 0 \leq x \leq y\} \supseteq [w, \bar{v}]$. So suppose $\eta > w$ and we construct a decreasing sequence $\{v_n\}$ with $v_0 = \eta$ by induction as follows. If $v_n \leq w$ then stop proceeding; otherwise let $v_{n+1} = \kappa(0, v_n, \delta)$. Notice that $v_{n+1} > w$ implies $v_{n+1} < v_n$ (for, $v_{n+1} \geq v_n$ implies $v_n \leq \underline{v}$), and, also, the construction stops by finite steps (otherwise the sequence converges to $\underline{v}$). Now for $w < v \in [v_{n+1}, v_n]$: if $n = 1$ then $v = \kappa(x, v_0, \delta)$ for some $x \in [0, y]$; if $n > 1$ then $v = \kappa(0, \zeta, \delta)$ for some $\zeta \in [v_n, v_{n-1}]$.

Given $A(.)$, let $\iota_s(z, \varsigma)$ be the same as in (20) except here $\iota_s(0, \varsigma) = 0$ (in (20) $\iota_s(0, \varsigma) = 0$ because there $y(\varsigma) = \alpha(\delta)[\bar{v}_s(\varsigma) - \underline{v}_s(\varsigma)]$). Let $\iota_b(z, \varsigma) = \iota_b(y(\varsigma), \varsigma)$ if $z \in (y(\varsigma), 1]$ and $\iota_b(z, \varsigma) = \varphi(z, y(\varsigma), \varphi_s(y(\varsigma), \delta), \delta)$ if $z \in [0, y(\varsigma)]$. Next we consider $y(\varsigma) > 0$ (treating $y(\varsigma) = 0$ as a trivial case). If $c \in [1 - y(\varsigma), 1)$, then by (39), $l_s'(y)[\alpha(\delta)u(y) + l_b(y)] = -l_b'(y)[l_s(y) - \alpha(\delta)y]$, where $y = 1 - c$, $l_b(x) = \varphi(x, y, \varphi_s(y, \delta), \delta)$, and $l_s(x) = \varphi_s(y, \delta) - \alpha(\delta)x$; by (41) ($x \mapsto l_b(x)$ is concave), $u'(y)[l_s(y) - \alpha(\delta)y] \geq \alpha(\delta)u(y) + l_b(y)$. So (y, y) solves the problem in (16) and $U_s(y, y, \varsigma) = \varphi_s(y(\varsigma), \delta) - \alpha(\delta)y(\varsigma)$. Also, if $c < 1 - y(\varsigma)$, then $(y(\varsigma), y(\varsigma))$ solves the problem. Finally, $(0, 0)$ solves the problem if $c = 0$ and $U_s(0, 0, \varsigma) = 0$.

Completion of the proof of Proposition 9

First we derive from (25)-(27) that

$$\frac{v_1 - v_{-N+1}}{\alpha(\delta)} = \frac{(K_1 - 0.5\rho)/K_2 - (\pi_1 + \pi_2)0.5\rho q^*}{1/(K_2\delta_N) + (\pi_1 + \pi_2)0.5(1-\rho)\delta}\delta, \tag{42}$$

$$\frac{v_2 - v_1}{\alpha(\delta)} = \frac{K_3K_1 + 0.5\rho q^* - K_3 0.5\rho}{1/(\pi_1 + \pi_2) + K_2K_3}\delta, \tag{43}$$

where $K_1 = \pi_1 0.5\rho u(q^*) + 0.5\rho$, $K_2 = 1 - 0.5(1-\rho)\delta\pi_1$, $K_3 = 0.5(1-\rho)\delta \cdot \delta_N$, and $\delta_N = \sum_{n=0}^{N-1} \delta^n$. By (25), $v_{-N+1} = 0.5\rho(1-\delta)\delta_N + \delta^N v_1$. Subtracting v_1 from both sides of this equality and from both sides of (27), we have

$$\frac{v_1 - v_{-N+1}}{1-\delta} = (v_1 - 0.5\rho)\delta_N, \ \frac{v_2 - v_1}{1-\delta} = (K_1 - v_1)/K_2. \tag{44}$$

Substituting $v_1 - v_{-N+1}$ and $v_2 - v_1$ in (44) into (26) and following some lengthy computation, we have

$$v_1 = \frac{K_1/K_2 - (\pi_1 + \pi_2)0.5\rho q^* + (\pi_1 + \pi_2)K_3 0.5\rho}{1/K_2 + (\pi_1 + \pi_2)K_3}. \tag{45}$$

Substituting (45) into (44) yields (42) and (43).

Next we construct $A(\varsigma)$ in the following nine cases.

(a) $v_b, v_s \le v_0$. Then $A(\varsigma) = (0, v_{b+}, v_{s+}, v_{b+}, v_{s+})$, where v_{b+} and v_{s+} satisfy $v_b = 0.5\rho(1-\delta) + \delta v_{b+}$ and $v_s = 0.5\rho(1-\delta) + \delta v_{s+}$.

(b) $v_b \le v_0$ and $v_s > v_0$. Then $A(\varsigma) = (0, v_{b+}, v_1, v_{b+}, v_1)$ for v_{b+} in (a).

(c) $v_b > v_0$ and $v_s \le v_0$. Then $A(\varsigma) = (0, v_1, v_{s+}, v_1, v_{s+})$ for v_{s+} in (a).

(d) $v_b, v_s > v_0$ and $v_b, v_s \notin V$. Then $A(\varsigma) = (0, v_b, v_s, v_b, v_s)$.

(e) $v_b > v_0$ and $v_s = v_2$. Then $A(\varsigma) = (0, v_1, v_1, v_1, v_1)$.

(f) $v_b = v \in (v_0, v_1)$ and $v_s = v_1$. Letting $\eta(x) = \varphi(x, q^*, v_1 - v_{-N+1}, \delta)$, then $A(\varsigma) = (\hat{y}(v), v_1, v_1, v_1 - \eta((\hat{y}(v)), v_{-N+1} + \alpha(\delta)(q^* - \hat{y}(v)))$. To define $\hat{y}(v)$, let $\lambda(x) = 0.5\pi_1\delta\{\rho\alpha(\delta)[u(q^*) - u(x)] + (1-\rho)[v_2 - v_1 + \eta(x)]\}$ and let $\lambda(\hat{y}(v)) = v_1 - v$. Existence of $\hat{y}(v)$ follows from continuity of $x \mapsto \lambda(x)$, $\lambda(q^*) = 0$, and $\lambda(0) = 0.5\pi_1\delta\{\rho\alpha(\delta)u(q^*) + (1-\rho)(v_2 - v_1)\} > v_1 - v_0$ (use (45), (25), and (27)).

(h) $v_b \ge v_1$ and $v_s = v_1$. Then $A(\varsigma) = (q^*, v_1, v_1, v_2, v_{-N+1})$.

(i) $v_b \in \{v_1, v_2\}$ and $v_s \in (v_0, v_1)$. Then $A(\varsigma) = (q^*, v_1, v_1, v_2, v_{-N+1})$.

(j) $v_b \in \{v_1, v_2\}$ and $v_s = v \in (v_1, v_2)$. Letting $\eta(x)$ be the same as in (f), then $A(\varsigma) = (\hat{y}(v), v_1, v_1, v_1 - \eta((\hat{y}(v)), v_{-N+1} + \alpha(\delta)(q^* - \hat{y}(v)))$. Here $\hat{y}(v)$ satisfies $0.5(\pi_1 + \pi_2)\delta[q^* - \hat{y}(v)] = v - v_1$.

To verify that (36) holds with the above constructed terms in the right side, compare it with (26) if $v \in (v_0, v_1)$ and with (27) if $v \in (v_1, v_2)$; v outside these two intervals should be evident.

Finally, we construct $h_{k,t}$ as follows. Refer to the proof of Proposition 6 and here we make four modifications. First let $h_{k,1}(\beta_0) = v_n$ if $k \in I_{0,n}$, where $I = I_{0,-N+1} \cup ... \cup I_{0,2}$ and the measure of $I_{0,n}$ is π_n. Second, if $v_b, v_s \notin \{v_n\}$, then let $g(r, \varsigma) = \varsigma$; otherwise, let $g(r, \varsigma) = (\bar{v}_b(\varsigma) + \iota_b(r_b^2, \varsigma), \underline{v}_s(\varsigma) + \iota_s(r_b^2, \varsigma))$. Third, let $D_t = \{i \in I : v_{i,t} \notin \{v_n\}\}$. Fourth, let $h_{k,t}(\beta_{t-1})$ be determined as follows.

(i) Either $\#D_t \leq 1$; or $D_t = \{i_1, i_2\}$ and $\phi_{i_1,t-1} = i_2$. Then $h_{k,t}(\beta_{t-1}) = v_{k,t}$.

(ii) Either $D_t = \{i_1, i_2\}$, $\phi_{i_1,t-1} \neq i_2$, $i_1 \in D_{t-1}$, and $i_2 \notin D_{t-1}$; or $D_t = \{i_1, i_2, i_3\}$ and $\phi_{i_2,t-1} = i_3$. If $k \in \{i_2, \phi_{i_2,t-1}\}$, then $h_{k,t}(\beta_{t-1}) = v_{k,t}$; otherwise $h_{k,t}(\beta_{t-1}) = h_{k,1}(\beta_0)$.

(iii) Other D_t. Then $h_{k,t}(\beta_{t-1}) = h_{k,1}(\beta_0)$.

The proof of Proposition 11

Suppose by contradiction that f is supported by an equilibrium σ for some trading mechanism. Let each of k and $j = \phi_{k,t}$ have some on-path start-of-t holdings. Let $\theta_{k,t} = 0$. Let l_n be the transfer of money specified by $\sigma_{k,t}$ and $\sigma_{j,t}$ when $\omega_{k,t} = n$, $n \in \{0, 1\}$. Then letting $v_i(l)$ be i's start-of-$t+1$ continuation value if l is the transfer of money at t, $v_i(l_1) = v_i(l_0) = w$, $i \in \{k, j\}$. Let $\underline{l} = \min\{l_1, l_0\}$ and $\bar{l} = \max\{l_1, l_0\}$. Free disposal of money implies $\underline{l} > 0$ (otherwise when $\omega_{k,t} = 1$, j is better off by choosing autarky and disposing of $-\underline{l}$ units of money). Then again by free disposal of money, $v_k(\bar{l}) = v_k(0)$ (otherwise when $\omega_{k,t} = 0$, k is better off by choosing autarky).

For each meeting between two agents with on-path holdings, there is such a $\bar{l}$. Put $\bar{l}$ from all such meetings together to form a set, called S. Now let the $\bar{l}$ at some date-t on-path meeting be sufficiently large in a way described below. For this meeting, call the buyer as k, denote his pre-meeting money holdings by $m_{k,t}$, and denote the $\bar{l}$ by $\bar{l}(t)$. Now we show that when $\omega_{k,t} = 0$, k can benefit from deviating to autarky.

With this deviation, k has $m_{k,t}$ at the start of $t+1$. If $\theta_{k,t+1} = 0$, then by free disposal of extra money k can keep his holdings on path before meeting $\phi_{k,t+1}$. So let $\theta_{k,t+1} = 1$ and let $l_n(t+1)$ be the equilibrium transfers of money if $\omega_{k,t+1} = n$ and if k carries $m_{k,t} - \bar{l}(t)$ into this date-t+1 meeting. If $\bar{l}(t+1) = \max\{l_1(t+1), l_0(t+1)\} \leq \bar{l}(t)$, then k can keep his holdings after the date-$t+1$ meeting on path (by free disposal of extra money) even though he deviates to autarky (so his consumption is 1 instead of $1-q$ when $\omega_{k,t+1}=1$) at $t+1$. Hence k's date-t deviation is beneficial when $\bar{l}(t)$ is such that the measure of meetings in $t+1$ with $\bar{l}(t+1) \leq \bar{l}(t)$ is sufficiently close to unity. If $\infty > \sup S \in S$, then let $\bar{l}(t) = \sup S$. If $\infty > \sup S \notin S$, then let $\bar{l}(t)$ be sufficiently close to $\sup S$. If $\sup S = \infty$, then let $\bar{l}(t)$ be sufficiently large so that the measure of agents with at least $\bar{l}(t)$ is sufficiently close to zero (recall that the sock of money is fixed).

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