An overlapping-generations model with search*

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Abstract

Search is embedded in an overlapping-generations model. The young participate in a centralized market, and then are matched in pairs in a decentralized market. The old only participate in the centralized market. If the buyer's bargaining power in pairwise trade is close to unity and if the old are risk averse, then the golden-rule rate of money transfer is positive. Such risk aversion, the pairwise meetings, and dependence of the young's saving on the rate of return are necessary for this result.

JEL classification: E30, E31, E40, E41

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1 Introduction

I develop a model of money with two frictions that usually do not appear in the same model: the overlapping-generations (OLG) friction and the search friction. In this model, people live for two periods. In each period, the young and the old participate in a centralized market; then the old die and the young are matched in pairs in a decentralized market.[1] This setup is tractable,

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[1] Maeda [9] and Russell [11] study different OLG models with search. In neither is there centralized trade.

because, as in any two-period-lived OLG model, the old sell all their money which implies that the inherited distribution of wealth induced by pairwise trade is not a state variable.[2] Moreover, the model can be specified so that the pairwise meetings are necessary for money to be valued.

There was a debate about whether the ordinary OLG model is a suitable model of money. Some argued that the model is misleading because of the absence of a transactions role of money (see, e.g., Tobin [13]). One response was to add cash-in-advance or money-in-the-utility-function (see, e.g., McCallum [10]). Here, a transactions role of money is incorporated using the pairwise meetings and anonymity.

This model has new implications for the optimal rate of lump-sum money creation compared to the ordinary OLG model and the OLG model with money-in-the-utility-function (see, e.g., Abel [1]); optimality here means maximization of the steady-state expected lifetime utility—golden-rule optimality. In particular, if the buyer's bargaining power in pairwise trade is close to unity and if the old are risk averse, then the golden-rule rate of money creation is positive. Such risk aversion, the pairwise meetings, and dependence of the young's saving on the rate of return are necessary for this result.

To see why a positive transfer can be optimal, let buyers have all the bargaining power and consider a small *reduction* in the zero-inflation steady state saving (in the form of money) of one young person. The first-order effect from this reduction on this person's own lifetime expected utility is zero. Now consider his partner in a pairwise meeting. If the partner is the seller in the meeting, then she receives the same utility from trading with this person and all other buyers. But if the trading partner is the buyer in the meeting, then she receives more utility from trading with this person than from other sellers for the following reason. Because the old are risk averse, the post-pairwise-meeting value function of money is strictly concave. And because this seller is poorer than other sellers, this strict concavity implies that he produces more than other sellers for the same payment. This leads to a better payoff for the trading partner. Therefore, there is a first-order effect from the person's reduced saving on the lifetime utility of others, an externality. Money creation that comes about through lump-sum transfers to the young produces such reduced saving (provided that saving depends on the rate of

[2] This feature is exploited by Zhu and Wallace [16, section 3] to complete a general-equilibrium analysis. The version there is special in that the young's saving is exogenous.

return) and, therefore, is beneficial. As the above explanation suggests, both risk aversion for the old and the pairwise meetings are necessary for this effect to occur.

The rest of the paper is organized as follows. In section 2, I describe the model and equilibrium and show existence. I establish results about the golden-rule rate of money transfer in section 3. In section 4, I discuss introducing capital into the model, and some insight from this model that may carry over to the model of Lagos and Wright [8].

2 The model

In this section, I describe the model, define equilibrium, and show existence.

2.1 Environment

Time is discrete, dated as $t \geq 0$. Each date has two stages, 1 and 2. At the start of each date t, there is a unit mass of newly-born people, and each person lives for three consecutive stages. The person is *young* at date t or his first two stages, and *old* at date $t+1$ or his third stage. There is one produced and perishable good per stage. There is another durable and intrinsically useless object called money; the initial old—those who die at the end of stage 1 of date 0—hold the initial money stock M_0.

A person born at $t \geq 0$ can produce but cannot consume at his first stage, and he can consume but cannot produce at his third stage. At his second stage, he has an equal chance to be a *buyer*—who can consume but cannot produce, or a *seller*—who can produce but cannot consume. At his ith stage, his utility from consuming q (given he can) is $u_i(q)$, and his disutility from producing is q (given he can) is $c_i(q)$. I assume throughout, unless specified otherwise, that $c_i' > 0$, $c_i'' \geq 0$, $u_i' > 0$, and $u_i'' \leq 0$. The person's lifetime utility is the sum of his stage utility, and he maximizes his expected lifetime utility.[3]

[3] In this setting, if the person's second stage is removed and $c_1(q) = U(\omega) - U(\omega - q)$ and $u_3(q) = V(e+q)$, then the model is equivalent to the ordinary two-period-lived OLG model in which each person is endowed with ω when young and e when old, and has the utility function U when young and V when old.

There are several variants of this setting. For example, the person can consume at his first stage and produce at his third stage. Also, the person can be idle at his second stage with some probability, and cannot produce at his third stage with some probability. Such

At stage 1 of date t, the young and old meet in a centralized spot market. At stage 2 of date t, the young are matched in pairs; matching is random but, without loss of generality, a buyer always meets a seller. People are anonymous so money has an essential role in facilitating trade. The trade in the centralized market is competitive. In each pairwise meeting, each person's money holding is common knowledge; the surplus from trade is split by generalized Nash bargaining in which the buyer's bargaining power is $\theta \in (0, 1]$. Finally, each young receives τM_t units of money from the government in the date t centralized market, where M_t is the average holding of the old at the start of date t and $\tau \geq \underline{\tau} > -1$ for some negative $\underline{\tau}$.[4]

2.2 Definition of equilibrium

Let ρ_t be the real balance held by the old (hereafter, the real balance when the context is clear) in the date t centralized market; that is, ρ_t is the product of M_t and the market price of money in term of goods. Let the individual state x be the ratio of the individual money holding to the current stock of money. Now I describe the individual choice problems by backward induction, starting with the end of date t.

First, a person who enters old age in state z will receive the payoff $u_3(z\rho_{t+1})$. Then consider a date t pairwise meeting between a buyer in state z_b and a seller in state z_s. A trade (q, l), where q is the transfer of the good and lM_{t+1} is the transfer of money, will give the buyer a surplus $u_2(q) + u_3(z_b\rho_{t+1} - l\rho_{t+1}) - u_3(z_b\rho_{t+1})$, and the seller a surplus $-c_2(q) + u_3(z_s\rho_{t+1} + l\rho_{t+1}) - u_3(z_s\rho_{t+1})$. Therefore, the trade reached in the meeting, denoted $(q(z_b, z_s; \rho_{t+1}), l(z_b, z_s; \rho_{t+1}))$, is a maximizer of the problem

$$\begin{aligned} \max_{q\geq 0, 0\leq l\leq z_b} & [u_2(q) + u_3(z_b\rho_{t+1} - l\rho_{t+1}) - u_3(z_b\rho_{t+1})]^{\theta} \\ & \times[-c_2(q) + u_3(z_s\rho_{t+1} + l\rho_{t+1}) - u_3(z_s\rho_{t+1})]^{1-\theta}. \end{aligned} \tag{1}$$

The buyer's payoff is

$$f(z_b, z_s; \rho_{t+1}) = u_2(q(z_b, z_s; \rho_{t+1})) + u_3(z_b\rho_{t+1} - l(z_b, z_s; \rho_{t+1})\rho_{t+1}), \tag{2}$$

and the seller's payoff is

$$g(z_b, z_s; \rho_{t+1}) = -c_2(q(z_b, z_s; \rho_{t+1})) + u_3(z_s\rho_{t+1} + l(z_b, z_s; \rho_{t+1})\rho_{t+1}). \tag{3}$$

variants are discussed in the working paper version [15]. The results are similar for all these variants.

[4]For a technical reason, I need τ in a closed set. But $\underline{\tau}$ can be arbitrarily close to -1.

Hence, before the pairwise meetings, the expected utility of a young person in state x is

$$v(x;\rho_{t+1},\mu_t) = 0.5\int [f(x,y;\rho_{t+1}) + g(y,x;\rho_{t+1})]\mu_t(dy), \tag{4}$$

where μ_t is the distribution of young individual states just before the pairwise meetings, and so the young person's problem in the date t centralized market is

$$\max -c_1((1+\tau)x\rho_t - \tau\rho_t) + v(x;\rho_{t+1},\mu_t) \text{ s.t. } x \geq \max\{0, \frac{\tau}{1+\tau}\}. \tag{5}$$

(When $\theta < 1$, f and g in (4) need not be concave in x, so v need not be concave in x. Hence the young need not leave the centralized market with the same amount of money and therefore μ_t need not be degenerate.) Let

$$X(\rho_t,\rho_{t+1},\mu_t) = \{x : x \text{ is a solution to (5)}\}. \tag{6}$$

The support of μ_t, denoted *supp* μ_t, must be a subset of X. Also, market clearing requires $\int x\mu_t(dx) = 1$. Therefore, we have

Definition 1 *A sequence $\{(\rho_t,\mu_t)\}_{t=0}^{\infty}$ is a monetary equilibrium given (θ,τ) if $\rho_t > 0$, $\int x\mu_t(dx) = 1$, and supp $\mu_t \subset X(\rho_t,\rho_{t+1},\mu_t)$, all t. A pair (ρ,μ) is a monetary steady state given (θ,τ) if the sequence $\{(\rho_t,\mu_t)\}_{t=0}^{\infty}$ with $(\rho_t,\mu_t) = (\rho,\mu)$, all t, is a monetary equilibrium.*

2.3 Existence of a monetary steady state

Existence can be established under fairly general assumptions.

Proposition 1 *Suppose $c_1'' > 0$, $c_1'(\infty) = \infty$, $c_2(q) = q$, $u_2(0) = 0$, $u_2'' < 0$, and $u_2'(0) = \infty$. Also, suppose $[u_2(q)/u_2'(q)][u_2''(q)/u_2'(q)](1-1/\theta)$ is bounded above for $q > 0$ in a neighborhood of 0. Then there exists a monetary steady state given (θ,τ).*

Proof. See the appendix. ■

The assumptions in Proposition 1 are maintained throughout unless explicitly noted. The boundedness condition on $h = (u_2/u_2')(u_2''/u_2')(1-1/\theta)$ holds if $\theta = 1$ and is otherwise satisfied if u_2 is a power function; $u_2(0) = 0$

and linearity of c_2 are without loss of generality; while $c_1'(\infty) = \infty$ ensures that the real balance is bounded above.

I prove Proposition 1 by constructing a mapping whose fixed point (ρ, μ) is a monetary steady state. As given in the appendix, this mapping assigns to each (ρ, μ) the set of probability measures implied by all randomizations over $X(\rho, \rho, \mu)$, and assigns to each measure σ in this set some ρ' such that $\rho' \gtrless \rho$ if $\int x\sigma(dx) \gtrless 1$. That is, the mapping raises (reduces) the real balance if the excess demand for money is positive (negative).

To carry out this approach, I must find a lower bound on ρ so that the excess demand for money is positive if ρ is close to this bound. In turn, I must show that for small ρ, the payoff to a buyer in a pairwise meeting from a marginal increment of his pre-meeting state $x \leq 1$ is sufficiently large. This uses $u_2'(0) = \infty$ and boundedness of h. I need to bound h because the best lower bound I can find on the buyer's payoff from such an increment is $[C + h(\rho)]^{-1}u_2'(\rho)u_3'(\rho)\rho$ for some constant C.

For comparison and for future reference, consider the model without the pairwise meetings; that is, the special case, $u_2 = c_2 = 0$. Then, we have

Corollary 1 *If $u_2 = c_2 = 0$, then $(1 + \tau)c_1'(0) < u_3'(0)$ is sufficient and necessary for existence of a monetary steady state under (θ, τ).*

Proof. Let (ρ, μ) be a monetary steady state with $u_2 = c_2 = 0$ given (θ, τ). Under the maintained assumptions, $\mu = \mu_d$, where *supp* $\mu_d = \{1\}$, and ρ is determined by $(1 + \tau)c_1'(\rho) = u_3'(\rho)$. Because $c_1'' > 0$, $(1 + \tau)c_1'(0) < u_3'(0) \Leftrightarrow \rho > 0$. ■

3 The optimal rate of money transfer

Here, optimality is maximization of the young person's steady-state expected lifetime utility, golden rule optimality. I give sufficient condition conditions for a positive transfer to be optimal.

Proposition 2 *If $u_3'(q)q$ is non decreasing and $u_3'' < 0$ and if θ is sufficiently close to 1, then the golden-rule rate of transfer is positive.*

Proof. The proof proceeds in two steps. In step 1, I prove the result for $\theta = 1$; a useful intermediate result is uniqueness of the monetary steady state under each τ. In step 2, I show that the result holds for θ near 1; here, I use the uniqueness result in step 1 and a continuity property of the main mapping in the proof of Proposition 1.

Step 1. As just indicated, $\theta = 1$ in this step. First, I characterize monetary steady states without appealing to monotonicity of $u_3'(q)q$ and $u_3'' < 0$. For each $\rho > 0$, let $(q(\rho), l(\rho))$ with $l(\rho) \in (0, 1]$ satisfy

$$q(\rho) = u_3(\rho + l(\rho)\rho) - u_3(\rho), \tag{7}$$

$$u_2'(q(\rho))u_3'(\rho + l(\rho)\rho) \geq u_3'(\rho - l(\rho)\rho) \text{ strict only if } l(\rho) = 1. \tag{8}$$

Notice that such $(q(\rho), l(\rho))$ is unique. For $\rho > 0$ and $\tau \geq \underline{\tau}$, let

$$F(\rho, \tau) = (1 + \tau)c_1'(\rho) - 0.5u_2'(q(\rho))u_3'(\rho + l(\rho)\rho) - 0.5u_3'(\rho). \tag{9}$$

I claim that (ρ, μ) is a monetary steady state under τ if and only if $\mu = \mu_d$ (*supp* $\mu_d = \{1\}$) and $F(\rho, \tau) = 0$, and that the young person's expected lifetime utility in the steady state (ρ, μ_d) is

$$W(\rho) = -c_1(\rho) + 0.5[u_2(q(\rho)) + u_3(\rho - l(\rho)\rho)] + 0.5u_3(\rho). \tag{10}$$

To see this claim, let (ρ, μ) be a monetary steady state given τ. By $\theta = 1$ and $c_2(q) = q$, the problem in (1) with $\rho_{t+1} = \rho$ can be written as

$$f(z_b, z_s; \rho) = \max u_2(q) + u_3(z_b\rho - l\rho) \text{ s.t. } q = u_3(z_s\rho + l\rho) - u_3(z_s\rho),\ 0 \leq l \leq z_b. \tag{11}$$

So $g(z_b, z_s; \rho) = u_3(z_s\rho)$, and $f(z_b, z_s; \rho)$ is concave in z_b. Hence $v(x; \rho, \mu)$ is concave in x and $\mu = \mu_d$. Now consider a young person in state x meets a partner in state 1. If this person is the seller, then $g(1, x; \rho) = u_3(x\rho)$. When $x = 1$, the payoff to this person from a marginal increment of his state is $u_3'(\rho)\rho$. If this person is the buyer, then by $u_2'(0) = \infty$, $(q(x, 1; \rho), l(x, 1; \rho))$ (the maximizer of the problem in (11) with $(z_b, z_s) = (x, 1)$) is the unique (q, l) satisfying $q = u_3(\rho + l\rho) - u_3(\rho)$, and $u_2'(q)u_3'(\rho + l\rho) \geq u_3'(x\rho - l\rho)$ strict only if $l = x$. By the envelope condition, when $x = 1$, the payoff to this person from a marginal increment of his state is $u_2'(q(\rho))u_3'(\rho + l(\rho)\rho)\rho$ (notice that $(q(\rho), l(\rho)) = (q(1, 1; \rho), l(1, 1; \rho))$). Hence $x = 1 \in X(\rho, \rho, \mu_d)$ (market clearing) if and only if $F(\rho, \tau) = 0$.

Next, I show a weaker result: *If* $c_1'(0) \geq u_3'(0)$, *then some positive transfer dominates zero transfer*. That is, when the pairwise meetings are necessary for money to be valued given $\tau = 0$ (see Corollary 1), some positive transfer is beneficial.

To see this, first fix $\tau \geq 0$. By Proposition 1, there exists some ρ, denoted $\rho(\tau)$, satisfying $F(\rho, \tau) = 0$. By $c_1'(0) \geq u_3'(0)$ and $\tau \geq 0$, $F(\rho, \tau) = 0$ only if the inequality in (8) is strict, or only if $G(\rho, \tau) = 0$, where

$$G(\rho, \tau) = (1 + \tau)c_1'(\rho) - 0.5u_2'(u_3(2\rho) - u_3(\rho))u_3'(2\rho) - 0.5u_3'(\rho).$$

By monotonicity of $u_3'(q)q$, for each $\rho > 0$, $G_1(\rho, \tau)$ (the derivative of $G(, .\tau)$ at ρ) is positive, so $\rho(\tau)$ is the unique ρ satisfying $G(\rho, \tau) = 0$. Hence $(\rho(\tau), \mu_d)$ is the unique monetary steady state given τ. Next, by the implicit function theorem (applied to $G(\rho(\tau), \tau) = 0$), for each $\tau \geq 0$, $\rho'(\tau) = -c_1'(\rho(\tau))/G_1(\rho(\tau), \tau) < 0$. Also, fixing $l(\rho) = 1$ in (10) and (7) and using $G(\rho(\tau), \tau) = 0$, we obtain the derivative of W at $\rho = \rho(\tau)$ as

$$W'(\rho) = \tau c_1'(\rho) + 0.5u_2'(q(\rho))\Delta,$$

where

$$\Delta = u_3'(2\rho) - u_3'(\rho).$$

By $u_3'' < 0$, $\Delta < 0$ so $V(\tau) = W(\rho(\tau))$ is strictly increasing on $[0, \epsilon]$ for some $\epsilon > 0$ (notice that $V'(0) = W'(\rho(0))\rho'(0) > 0$). This proves the weaker result.

Next, I show that the golden rule rate of transfer is positive. Now I have to directly work on F. A technical issue is that $F_1(\rho, \tau)$ (the derivative of $F(., \tau)$ at ρ) need not always be defined. This issue is dealt with in the appendix. By Lemma 3 (which applies if $u_3'(q)q$ is non decreasing and $u_3'' < 0$, or if $u_3'' = 0$), for each $\tau \geq \underline{\tau}$, there exists a unique ρ, denoted $\rho(\tau)$, satisfying $F(\rho, \tau) = 0$; the derivative of W at $\rho = \rho(\tau)$ is

$$W'(\rho) = \tau c_1'(\rho) + 0.5u_2'(q(\rho))\Delta, \tag{12}$$

where

$$\Delta = u_3'(\rho + l(\rho)\rho) - u_3'(\rho);$$

and the left and right derivatives of $\rho(.)$ at τ (the right derivative if $\tau = \underline{\tau}$) are defined and negative. By $u_3'' < 0$, $\Delta < 0$ so $V(\tau) = W(\rho(\tau))$ is strictly increasing on $[\underline{\tau}, \epsilon]$ for some $\epsilon > 0$. This completes step 1.

Step 2. See Lemma 4 in the appendix. ■

By the definition of W' (see (12)), in the steady state $(\rho(\tau), \mu_d)$, if the real balance changes by $d\rho$, then the young person's expected lifetime utility changes by $W'd\rho$. Because $W' \neq 0$, there are effects from the change in the real balance the young person can not capture by his own action—in specific, by choosing his own saving in term of money (the young person's expected lifetime utility changes by $F(\rho(\tau), \tau)d\rho$ if he changes his real saving by $d\rho$). In (12), $\tau c_1'(\rho)$ reflects one such effect: The young person's real wealth changes by $\tau d\rho$ in a lump sum way. Also, $u_2'(q(\rho))\Delta$ reflects another such effect: When the young person is the buyer in a pairwise meeting, if his payment measured by its real value in the coming competitive market does not change, then the amount of good he receives from the seller changes by $\Delta d\rho$. The first effect presents in ordinary OLG models. The second does not; it comes from the externality described in the introduction.

There is no such externality when the old are risk neutral, so a positive transfer cannot dominate zero transfer.

Corollary 2 *If $u_3'' = 0$ and if θ is sufficiently close to 1, then the golden-rule rate of transfer is non positive.*

Proof. When $\theta = 1$, $u_3'' = 0$ implies $\Delta = 0$ in (12). So the golden-rule rate of transfer is zero. The rest of the proof follows from the exact argument in the proof of Lemma 4. ■

I suspect that Corollary 2 holds for general $\theta < 1$, but proving such result is difficult because the distribution μ in a monetary steady state need not be degenerate. With more restrictive assumptions to ensure such degeneracy for general θ, I can show that $\rho(\tau)$ is strictly decreasing in τ and $W'(\rho(0))$ has the same sign as

$$\eta \equiv (1-\theta) + (1-\theta)u_2'(q(1,1;\rho)) + \theta u_2''(q(1,1;\rho))[q(1,1;\rho) - \rho], \qquad (13)$$

which implies some negative transfer dominates zero transfer ($\theta < 1 \Rightarrow q(1,1;\rho) < \rho \Rightarrow \eta > 0$). The term η reflects the so-called holdup problem. This problem does not rely on the curvature of u_3, and it can be dominant when the buyer's bargaining power is far away from unity so that some negative transfer dominates zero transfer even if the old are risk averse.

When the pairwise meetings are dropped, both the externality driving Proposition 2 and the holdup problem are absent and there is no room for policy intervention.

Corollary 3 *If $u_2 = c_2 = 0$, then the golden-rule rate of money transfer is zero.*

Proof. If $c_1'(0) \geq u_3'(0)$, then $\rho > 0$ only if $\tau < 0$ (see Corollary 1). But the young person's expected steady state utility is less than $a = -c_1(0) + u_3(0)$. If $c_1'(0) < u_3'(0)$ and if $(1+\tau)c_1'(\rho) = u_3'(\rho)$ has a positive solution, denoted $\rho(\tau)$, then the young person's expected lifetime utility is $W(\rho(\tau)) = -c_1(\rho(\tau)) + u_3(\rho(\tau))$ (notice that $W(\rho(0)) > a$). Because $W'(\rho(\tau)) = \tau c_1'(\rho(\tau))$ and $\rho'(\tau) < 0$, the golden-rule rate is zero. ■

Policy is irrelevant if the real balance does not respond to it.

Corollary 4 *If $c_1(q) = 0$ for $q \in [0, \omega]$ and $c_1(\omega) = \infty$ for $q > \omega$, then for any θ, the young person's steady-state expected lifetime utility does not depend on τ.*

Proof. Fix θ. It suffices to note that under each τ, there exists a unique monetary steady state $(\rho, \mu) = (\omega, \mu_d)$. ■

Thus far, I consider transfers to the young in the centralized market. Such transfers and transfers to the old are equivalent if the cash constraints of buyers are not binding in the pairwise meetings. Otherwise, the two types of transfers support different allocations and the optimality of a positive transfer to the young does not imply the optimality of a positive transfer to the old. For example, in a Corollary 4 setting with $\omega = 1$, $u_2(q) = 2\sqrt{q}$, and $u_3(q) = q$, all transfers to the young are equivalent, but any positive transfer to the old decreases welfare of the young. A related point is the risk-sharing effect of money creation. Levine [7] presents an infinitely-lived agents model in which money creation is beneficial because of this effect. Pairwise trade in this OLG model does induce a risk to each young person, but, by itself, the risk does not call for policy intervention. Indeed, in any Corollary 4 setting, a positive transfer to the old gives rise to risk-sharing, but in the above example such a transfer reduces the young person's steady-state expected lifetime utility.

Finally, even if the pairwise trading protocol is price taking, a positive transfer to the young can still be beneficial. With some additional assumptions, I can show $\rho'(\tau) < 0$ and $W'(\rho(0))$ has the same sign as $(\kappa/\rho)'$, where κ is the stage 2 real balance. Moreover, $(\kappa/\rho)' < 0$ if $u_3'' < 0$, and $(\kappa/\rho)' = 0$ if $u_3'' = 0$. So some positive transfer dominates zero transfer if $u_3'' < 0$. To see why $(\kappa/\rho)'$ matters, consider a buyer in state 1 who chooses q_{b2} (his stage 2 consumption) and q_{b3} (his stage 3 consumption) to maximize

$$u_2(q_{b2}) + u_3(q_{b3}), \text{ s.t. } (\kappa/\rho)q_{b3} + q_{b2} = \kappa \text{ and } q_{b2} \leq \kappa.$$

Now a change in ρ has two two effects on his first constraint—one through a change in κ and the other through a change in (κ/ρ), and the second effect is the one that he can not capture by his own action.

4 Discussions

There is no capital in the present model. One may introduce capital as an input for production in the centralized market. To maintain the transactions role of money, one may assume that neither capital nor certificates of its ownership can be carried into the pairwise meetings.[5] More interestingly, one could assume some private information about the quality of capital, similar to the private information on the quality of goods in Williamson and Wright [14]. Neither approach would eliminate the externality driving Proposition 2. Of course, in a version with capital, money creation could induce more capital accumulation, an effect that has long been discussed in literature (see, e.g., Tobin [12]).

Deviatov and Wallace [6] and Deviatov [5] study a search model with infinitely-lived agents and with only pairwise trade. They select a trading protocol to maximize a social planner's objective function. They find examples in which money creation is beneficial. Results from [5, 6] motivate the following question: In the OLG model, if the government can choose the rate of money transfer and the trading protocol (subject to some incentive constraints), does the optimal policy involve money creation?

Lagos and Wright [8] (LW) build an infinitely-lived agent model with alternating decentralized-centralized markets. For tractability, they assume quasi-linear preferences on centralized-trade goods. This special assumption implies the post-pairwise-meeting value function of money is affine. They find that the Friedman rule is optimal. This OLG model was initially motivated to understand significance of wealth effects of money creation that are eliminated by their special assumption. There is a sort of observational equivalence between the LW model and the OLG model with the risk neutral old. While the effects from money creation differ in the LW model and the equivalent OLG model, the difference disappears if people in the LW model are not endowed with the initial money stock (see the appendix for detail). Based on these observations and Proposition 2, I suspect that the Friedman

[5] Aruoba and Wright [3] make such an assumption. As long as olds are risk averse, one can break dichotomy reported by [3] in the OLG model.

rule is not optimal in the LW model if quasi linearity is replaced by strict concavity and if the buyer's bargaining power is close to unity.[6]

This OLG model is tractable because people only live for three stages. Such a model is not suitable for serious quantitative exercises, but it has a natural extension for such exercises. One version would have, an old person dies with some probability. Of course, then, the inherited distribution of wealth would become a state variable. Tractability would be lost in such a version model, so it would have to be solved numerically.

Appendix

Proof of Proposition 1

In this proof, I assume that $u_3'(0)$ is finite, the case of the most interest. (When $u_3'(0) = \infty$, the proof is similar but differs in some detail.) Let $\rho_0 > 0$ and $\rho_1 > \rho_0$, and let $Z > 1$. Let $\boldsymbol{\rho} = [\rho_0, \rho_1]$, and let $\boldsymbol{\mu}$ be the set of all probability measures on $[0, Z]$. For $\mu \in \boldsymbol{\mu}$, let $E_\mu \equiv \int x\mu(dx)$, and let the measure $\bar{\mu}$ be defined by $\bar{\mu}(A)E_\mu = \mu(A)$ if $E_\mu > 1$, and $\bar{\mu}(A) = \mu(A)$ if $E_\mu \leq 1$. For $(\rho, \mu) \in \boldsymbol{\rho} \times \boldsymbol{\mu}$, let $\Sigma(\rho, \mu)$ be the set of probability measures implied by all randomizations over $X(\rho, \rho, \bar{\mu})$ (see (6)). Then let

$$T(\rho, \mu) = \{(\phi(\rho, \sigma), \sigma) : \sigma \in \Sigma(\rho, \mu)\} \tag{14}$$

$$\phi(\rho, \sigma) = \rho + \rho_0(E_\sigma - 1)/Z. \tag{15}$$

A fixed point of T is a monetary steady state under (θ, τ).

The next lemma provides conditions that ensures existence of a fixed point of T.

Lemma 1 *Let $\rho_0 \in (0, 1)$, $\rho_1 > 3$, and $Z > 1$. Suppose for $(\rho, \mu) \in \boldsymbol{\rho} \times \boldsymbol{\mu}$, (R1) $\rho > \rho_1 - \rho_0 \Rightarrow \max X(\rho, \rho, \bar{\mu}) \leq 1$; (R2) $\rho < 2\rho_0 \Rightarrow \min X(\rho, \rho, \bar{\mu}) \geq 1$; and (R3) $\max X(\rho, \rho, \bar{\mu}) \leq Z$. Then there exists a fixed point of T.*

Proof. First, fix $(\rho, \mu) \in \boldsymbol{\rho} \times \boldsymbol{\mu}$ and $\sigma \in \Sigma(\rho, \mu)$. Let $\rho' = \rho + \rho_0(E_\sigma - 1)/Z$. Now $\rho \in [2\rho_0, \rho_1 - \rho_0]$ and (R3)$\Rightarrow \rho' \in \boldsymbol{\rho}$; $\rho > \rho_1 - \rho_0$ and (R1)$\Rightarrow E_\sigma \leq 1 \Rightarrow \rho' \in \boldsymbol{\rho}$; and $\rho < 2\rho_0$ and (R2)$\Rightarrow E_\sigma \geq 1 \Rightarrow \rho' \in \boldsymbol{\rho}$. So $\phi(\rho, \sigma) \subset \boldsymbol{\rho}$. Then by (R3), $T(\rho, \mu) \subset \boldsymbol{\rho} \times \boldsymbol{\mu}$.

[6] This conjecture seems to be consistent with the result reported by Chiu and Molico [4] who study such a LW model. That is, the welfare costs of inflation in their model are much lower than those in [8]. The externality driving Proposition 2 seems to contribute to this result.

Next, by $u_3'' \leq 0$ and $u_2'' < 0$, $(q(x,y;\rho), l(x,y;\rho))$ (see (1)) is the unique (q,l) satisfying

$$\begin{aligned} & u_2'(q)u_3'(y\rho + l\rho) \geq u_3'(x\rho - l\rho) \text{ strict only if } l = x, && (16) \\ & \theta u_2'(q)[-q + u_3(y\rho + l\rho) - u_3(y\rho)] && (17) \\ = \; & (1-\theta)[u_2(q) + u_3(x\rho - l\rho) - u_3(x\rho)]. \end{aligned}$$

Then by the theorem of maximum (see [2, p. 473])), $q(.,.;.)$ and $l(.,.;.)$ are uniformly continuous on $[0,Z]^2 \times \boldsymbol{\rho}$ (uniformly because the domain is compact), so are $f(.,.;.)$ and $g(.,.;.)$ (see (2)-(3)). Let $\mathbf{V}$ be the set of all non decreasing and continuous real-valued functions on $[0,Z]$. Fix $(\rho,\mu) \in \boldsymbol{\rho} \times \boldsymbol{\mu}$. It is standard to show (by $u_2'' < 0$ and $u_3' \leq 0$ and (16)-(17)) that for each $y \in [0,Z]$, $f(.,y;\rho)$ and $g(y,.;\rho)$ are strictly increasing. So $v(.;\rho,\bar{\mu})$ (see (4)) is strictly increasing. Because $f(.,.;\rho)$ and $g(.,.;\rho)$ are uniformly continuous on $[0,Z]^2$, by [2, 12.6 Corollary, p. 417], $x_n \to x$ implies $v(x_n;\rho,\bar{\mu}) \to v(x;\rho,\bar{\mu})$. So $v(.;\rho,\bar{\mu})$ is continuous. Hence $v(.;\rho,\bar{\mu}) \in \mathbf{V}$.

Next, let $\mathbf{V}$ be equipped with the sup norm topology. Let $\boldsymbol{\mu}$ be equipped with the weak* topology. By [2, 12.10 Theorem, p. 419], $\boldsymbol{\mu}$ is metrizable and compact. Fix $x \in [0,Z]$. Because $f(x,.;.)$ and $g(.,x;.)$ are uniformly continuous on $[0,Z] \times \boldsymbol{\rho}$, by [2, 12.6 Corollary, p. 417], $(\rho_n,\mu_n) \to (\rho,\mu)$ (so $\bar{\mu}_n \to \bar{\mu}$) implies $v(x;\rho_n,\bar{\mu}_n) \to v(x;\rho,\bar{\mu})$. Because x is arbitrary and because $v(.;\rho,\bar{\mu}), v(.;\rho_n,\bar{\mu}_n) \in \mathbf{V}$, it follows that $v(.;\rho_n,\bar{\mu}_n) \to v(.;\rho,\bar{\mu})$. Hence $(\rho,\mu) \mapsto v(.;\rho,\bar{\mu})$ is continuous. Because $k : [0,Z] \times \boldsymbol{\rho} \times \mathbf{V} \to \mathbb{R}$ with $k(z,\rho,v) = -c_1((1+\tau)z\rho - \tau\rho) + v(z)$ is continuous, by the theorem of maximum, $Y : \boldsymbol{\rho} \times \mathbf{V} \to [0,Z]$ with $Y(\rho,v) = \arg\max_z k(z,\rho,v)$ where $z \in [\max\{0, \frac{\tau}{1+\tau}\}, Z]$ is compact-valued and upper hemicontinuous. Hence $(\rho,\mu) \mapsto X(\rho,\rho,\bar{\mu})$ (the composition of $(\rho,\mu) \mapsto v(.;\rho,\bar{\mu})$ and $(\rho,v) \mapsto Y(\rho,v)$) is compact-valued and u.h.c. on $\boldsymbol{\rho} \times \boldsymbol{\mu}$; so is $(\rho,\mu) \mapsto \Sigma(\rho,\mu)$; and so is $(\rho,\mu) \mapsto T(\rho,\mu)$. Because Σ and hence T is convex-valued, by Fan's fixed point theorem, there exists a fixed point of T. ■

The next lemma completes the proof.

Lemma 2 *There exist $\rho_0 \in (0,1)$, $\rho_1 > 3$, and $Z > 1$ satisfying (R1)-(R3) in Lemma 1.*

Proof. The proof proceeds by three steps. In step 1, I show that there exists $\rho_1 > 3$ satisfying (R1); in step 2, I show that there exists $\rho_0 \in (0,1)$

satisfying (R2); in step 3, I show that given (ρ_0, ρ_1), there exists $Z > 1$ satisfying (R3). In this proof, I suppress dependence of $q(x, y; \rho)$, $l(x, y; \rho)$, $f(x, y; \rho)$, $g(x, y; \rho)$, and $v(x; \rho, \bar{\mu})$ on (ρ, μ).

Step 1. Let $\tilde{\rho} > 2$ satisfy $u_2'(\delta) < 1$, where $u_2(\delta) = u_3(\tilde{\rho}) - u_3(1)$, and $(1+\tau)c_1'(\tilde{\rho}) > u_3'(1)$. Then let $\rho_1 > \tilde{\rho} + 1$. Fix μ, $\rho > \rho_1 - 1$ and $z > 1$. It is standard to show $g(y, z) - g(y, z - \epsilon) \le u_3'(z\rho - \epsilon\rho)\epsilon\rho$, $\forall\ \epsilon \in (0, z)$ and y (by $u_2'' < 0$ and $u_3' \le 0$ and (16)-(17)). Now I claim $f(z, y) - f(z - \epsilon, y) \le u_3'(1)\epsilon\rho$, $\forall\ \epsilon \in (0, z-1)$ and y. Therefore, $u_3'(1)\epsilon\rho \ge v(z) - v(z - \epsilon)$ for small ϵ. Then by $a = [(1+\tau)z - \tau]\rho > \tilde{\rho}$ and $(1+\tau)c_1'(\tilde{\rho}) > u_3'(1)$, $v(z) - v(z-\epsilon) < c_1(a) - c_1(a - (1+\tau)\epsilon\rho)$ for small ϵ, so $z \notin X(\rho, \rho, \bar{\mu})$ and hence $\max X(\rho, \rho, \bar{\mu}) \le 1$.

For the claim, fix $x > 1$ and y, and it suffices to show $f_1(x, y) \le u_3'(1)\rho$ ($f_1(x, y)$ is the derivative of $f(., y)$ at x). Let $(\kappa, \iota) = (q(x, y), l(x, y))$. Suppose $x\rho - \iota\rho < 1$. Then by $x\rho > \tilde{\rho}$ and $u_2(\kappa) > u_3(x\rho) - u_3(x\rho - \iota\rho)$, $\kappa > \delta$ so $u_2'(\kappa) < 1$. Then by (16), $\iota < 0.5x$ so $0.5x\rho < x\rho - \iota\rho < 1$ or $x\rho < 2$, which contradicts $\tilde{\rho} > 2$. So $x\rho - \iota\rho \ge 1$ and (16) holds with equality. Then by the implicit function theorem (applied to (16) with equality and (17)), $q_1(x, y)$ (the derivative of $q(., y)$ at x) and $l_1(x, y)$ (the derivative of $l(., y)$ at x) are defined, in particular,

$$q_1 = \frac{\theta u_2'(\kappa)u_3'(y\rho + \iota\rho)l_1 - (1-\theta)u_3'(x\rho - \iota\rho)(1 - l_1) + (1-\theta)u_3'(x\rho)}{u_2'(\kappa) - (1-\theta)[u_2(\kappa) + u_3(x\rho - \iota\rho) - u_3(x\rho)]u_2''(\kappa)/u_2'(\kappa)}\rho.$$

So $f_1(x, y)$ is defined, in particular, $f_1 = u_2'(\kappa)q_1 + u_3'(x\rho - \iota\rho)(1 - l_1)\rho$. It is standard to show $l_1 \le 1$ (by $u_2'' < 0$ and $u_3' \le 0$ and (16)-(17)). Then by $x\rho - \iota\rho \ge 1$, $f_1(x, y) \le u_3'(1)\rho$.

Step 2. Let $h = (u_2/u_2')(u_2''/u_2')(1 - 1/\theta)$. Let $M > 0$ and $Q > 0$ satisfy $1 + 1/\theta + h(q) < M$, $\forall\ 0 < q \le Q$. Let $\bar{\rho} \in (0, 1)$ satisfy $\bar{q} = u_3(\bar{\rho}) - u_3(0) \le Q$ and $u_2'(\bar{q})u_3'(3\bar{\rho}) > \max\{u_3'(0), 4M(1+\tau)c_1'(\bar{\rho})\}$. Then let $\rho_0 \in (0, 0.5\bar{\rho})$. Fix μ, $\rho \in (0, 2\rho_0)$ and $z < 1$. Notice that $f(z+\epsilon, y) > f(z, y)$ and $g(y, z+\epsilon) > g(y, z)$, $\forall\ \epsilon > 0$ and y (see the proof of Lemma 1). Now I claim $M[f(z+\epsilon, y) - f(z, y)] > u_2'(\bar{q})u_3'(3\bar{\rho})\epsilon\rho$, $\forall\ \epsilon \in (0, 1-z)$ and $y \le 2$. Therefore, by $\bar{\mu}\{y : y \le 2\} \ge 1/2$, $4M[v(z+\epsilon) - v(z)] > u_2'(\bar{q})u_3'(3\bar{\rho})\epsilon\rho$ for such ϵ. Then by $a = [(1+\tau)z - \tau]\rho < \bar{\rho}$ and $u_2'(\bar{q})u_3'(3\bar{\rho}) > 4M(1+\tau)c_1'(\bar{\rho})$, $v(z+\epsilon) - v(z) > c_1(a + \epsilon(1+\tau)\rho) - c_1(a)$ for small ϵ, so $z \notin X(\rho, \rho, \bar{\mu})$ and hence $\min X(\rho, \rho, \bar{\mu}) \ge 1$.

For the claim, fix $x < 1$ and $y \le 2$, and it suffices to show $Mf_1(x, y) > u_2'(\bar{q})u_3'(3\bar{\rho})\rho$. By $u_2'(\bar{q})u_3'(3\bar{\rho}) > u_3'(0)$ and $\rho < \bar{\rho}$, $l(x, y) = x$ so $l_1(x, y) = 1$, and $q(x, y)$ is the q satisfying (17) when $l = x$. By the implicit function theorem, $q_1(x, y)$ is defined and so is $f_1(x, y)$; in particular, letting $(\kappa, \iota) =$

$(q(x,y), l(x,y))$, q_1 and f_1 are determined by the formulas given in step 1. Then by $\kappa < \bar{q}$ and $l_1 = 1$, $Mf_1(x,y) > u_2'(\bar{q})u_3'(3\bar{\rho})\rho$.

Step 3. Given ρ_0, let $Z > 1$ satisfy $(1+\tau)c_1'((1+\tau)Z\rho_0 - \tau\rho_0) > u_3'(1)$. Fix μ, $\rho \geq \rho_0$ and $z > Z$. By the argument in step 1, I can show $z \notin X(\rho, \rho, \bar{\mu})$, so $\max X(\rho, \rho, \bar{\mu}) \leq Z$. ■

Completion of the proof of Proposition 2

The next lemma completes the proof of step 1.

Lemma 3 *If either $u_3'(q)q$ is non decreasing and $u_3'' < 0$, or if $u_3'' = 0$, then the following is true when $\theta = 1$. (i) For each $\tau \geq \underline{\tau}$, there exists a unique ρ, denoted $\rho(\tau)$, satisfying $F(\rho, \tau) = 0$. (ii) $\rho(.) : [\underline{\tau}, \infty) \to \mathbb{R}$ is strictly decreasing. (iii) For each $\tau \geq \underline{\tau}$, the left and right derivatives of $\rho(.)$ at τ (the right derivative if $\tau = \underline{\tau}$) are defined and negative. (iv) For each $\tau \geq \underline{\tau}$, the derivative of W at $\rho = \rho(\tau)$ is $W'(\rho) = \tau c_1'(\rho) + 0.5u_2'(q(\rho))[u_3'(\rho + l(\rho)\rho) - u_3'(\rho)]$.*

Proof. To see the technical issue indicated in the main text, let (ρ, μ_d) be a monetary steady state given τ. When $c_1'(0) < u_3'(0)$ or $\tau < 0$, it may be the case that $l(\rho) = 1$ and (8) holds with equality. Now $l'(\rho)$ need not be defined, so $F_1(\rho, \tau)$ need not be defined (this issue does not depend on the curvature of u_3). To deal with it, I first establish some preliminary results that are organized in five claims.

Claim 1: There exists at most one $\hat{\rho}$ satisfying

$$u_2'(u_3(2\hat{\rho}) - u_3(\hat{\rho}))u_3'(2\hat{\rho}) = u_3'(0). \tag{18}$$

To see this, first notice that $u_2'' < 0$. If $u_3'' = 0$, then the result follows immediately; if $u_3'' < 0$, then the result follows from $2u_3'(2\rho) - u_3'(\rho) \geq 0$.

Claim 2: If $\rho \leq \hat{\rho}$, then $l(\rho) = 1$. To see this, first notice that $l(\hat{\rho}) = 1$. Next let $\rho < \hat{\rho}$, and suppose $l(\rho) < 1$ so (8) holds with equality. If $u_3'' = 0$, then this equality contradicts (18); if $u_3'' < 0$, then by $u_3(\rho + l(\rho)\rho) - u_3(\rho) \leq u_3(2\hat{\rho}) - u_3(\hat{\rho})$, this equality contradicts (18).

Claim 3: If $\rho > \hat{\rho}$, then $l(\rho) < 1$ and (8) holds with equality. To see this, notice that $u_2'(u_3(2\rho) - u_3(\rho))u_3'(2\rho) \geq u_3'(0)$ and $\rho > \hat{\rho}$ contradict (18).

Claim 4: $l'(\rho)$ is defined at $\rho \neq \hat{\rho}$; also, $l_-'(\hat{\rho})$ (the left derivative of $l(.)$ at $\hat{\rho}$) and $l_+'(\hat{\rho})$ (the right derivative of $l(.)$ at $\hat{\rho}$) are defined. To see this, first by claim 2, $l'(.)$ is defined over $(0, \hat{\rho}]$, where $l'(\hat{\rho}) = l_-'(\hat{\rho})$. Next by claim 3 and the implicit function theorem (applied to (8) with equality), $l'(.)$ is defined over $[\hat{\rho}, \infty)$, where $l'(\hat{\rho}) = l_+(\hat{\rho})$.

Claim 5: For each $\tau \geq \underline{\tau}$, $F_1(\rho, \tau)$ is defined and positive at $\rho \neq \hat{\rho}$; also, $F_{1-}(\hat{\rho}, \tau)$ (the left derivative of $F(., \tau)$ at $\hat{\rho}$) and $F_{1+}(\hat{\rho}, \tau)$ (the right derivative of $F(., \tau)$ at $\hat{\rho}$) are defined and positive. To see this, fix τ. First by claims 2 and 4, $F_1(., \tau)$ is defined over $(0, \hat{\rho}]$ and is positive valued, where $F_1(\hat{\rho}, \tau) = F_{1-}(\hat{\rho}, \tau)$. Next by claims 1, 3 and 4, $F_1(., \tau)$ is defined over $[\hat{\rho}, \infty)$, where $F_1(\hat{\rho}, \tau) = F_{1+}(\hat{\rho}, \tau)$, and to see $F_1(\rho, \tau) > 0$, if $l'(\rho) \geq 0$, then directly differentiate (9) w.r.t. ρ; otherwise, first substitute (8) with equality into (9) and then differentiate it w.r.t. ρ.

Now part (i) follows from claim 5. Part (ii) follows from claim 5 and $F(\rho(\tau), \tau) = 0$.

For part (iii), first by part (ii), there exists at most one $\hat{\tau}$ satisfying $\rho(\hat{\tau}) = \hat{\rho}$. Then by claims 2 and 3, $l(\rho(\tau)) < 1$ if $\tau < \hat{\tau}$, and $l(\rho(\tau)) = 1$ if $\tau \geq \hat{\tau}$. Now by claim 5 and the implicit function theorem (applied to $F(\rho(\tau), \tau) = 0$), $\rho'(.)$ is defined over $[\underline{\tau}, \hat{\tau}]$, where $\rho'(\hat{\tau}) = \rho'_-(\hat{\tau})$ (the left derivative of $\rho(.)$ at $\hat{\tau}$); also, $\rho'(.)$ is defined over $[\hat{\tau}, \infty)$, where $\rho'(\hat{\tau}) = \rho'_+(\hat{\tau})$ (the right derivative of $\rho(.)$ at $\hat{\tau}$). In specific, $\rho'(\tau) = -c_1'(\rho(\tau))/F_1(\rho(\tau), \tau)$ at $\tau \neq \hat{\tau}$, $\rho'_-(\hat{\tau}) = -c_1'(\hat{\rho})/F_{1+}(\hat{\rho}, \hat{\tau})$, and $\rho'_+(\hat{\tau}) = -c_1'(\hat{\rho})/F_{1-}(\hat{\rho}, \hat{\tau})$.

For part (iv), because $l(\rho) = 1$ for $\rho \leq \hat{\rho}$ and $F(\hat{\rho}, \hat{\tau}) = 0$, the left derivative of W at $\hat{\rho}$ is $\hat{\tau} c_1'(\hat{\rho}) + 0.5 u_2'(q(\hat{\rho}))[u_3'(2\hat{\rho}) - u_3'(\hat{\rho})]$. Because (8) holds with equality for $\rho \geq \hat{\rho}$ and $F(\hat{\rho}, \hat{\tau}) = 0$, the right derivative of W at $\hat{\rho}$ is $\hat{\tau} c_1'(\hat{\rho}) + 0.5 u_2'(q(\hat{\rho}))[u_3'(\hat{\rho} + l(\hat{\rho})\hat{\rho}) - u_3'(\hat{\rho})]$. By $l(\hat{\rho}) = 1$, the two derivatives are equal. Evidently, $W'(\rho)$ is defined at $\rho \neq \hat{\rho}$ and takes the purported value. ■

The next lemma deals with step 2 of the proof.

Lemma 4 *There exists some $\bar{\theta} < 1$ such that the golden rule rate of transfer is positive if $\theta > \bar{\theta}$.*

Proof. First, adding θ into the lists of arguments of $q(.)$, $l(.)$, $f(.)$, and $g(.)$, and denoting a generic (x, y, ρ, θ) by a, we have mappings $a \mapsto q(a)$, $a \mapsto l(a)$, $a \mapsto f(a)$, and $a \mapsto g(a)$. By the theorem of maximum, $q(.)$ and $l(.)$ are continuous at any $a = (x, y, \rho, \theta)$ with $\theta < 1$. Now let $a_n = (x_n, y_n, \rho_n, \theta_n) \rightarrow \hat{a} = (\hat{x}, \hat{y}, \hat{\rho}, 1)$. Let $(\hat{q}, \hat{l})$ be an arbitrary limit point of $\{(q(a_n), l(a_n))\}$. Because (16)-(17) hold when $(q, l, a) = (q(a_n), l(a_n), a_n)$, it follows that (16)-(17) hold when $(q, l, a) = (\hat{q}, \hat{l}, \hat{a})$. Because $(q(\hat{a}), l(\hat{a}))$ is the unique (q, l)

satisfying (16)-(17) when $a = \hat{a}$, it follows that $(\hat{q}, \hat{l}) = (q(\hat{a}), l(\hat{a}))$. Hence $q(.)$ and $l(.)$ are continuous, and so are $f(.)$ and $g(.)$. Also, adding (θ, τ) into the list of arguments of $v(.)$, we have mappings $(x, \rho, \mu, \theta, \tau) \mapsto v(x, \rho, \bar{\mu}, \theta, \tau)$, $(\rho, \mu, \theta, \tau) \mapsto X(\rho, \bar{\mu}, \theta, \tau)$, and $(\rho, \mu, \theta, \tau) \mapsto T(\rho, \mu, \theta, \tau)$ (for T, see (14) and (15)).

In what follows, let $(\rho, \mu, \theta, \tau)$ stand for the monetary steady state (ρ, μ) under (θ, τ), and let $W(\rho, \mu, \theta, \tau)$ be the young's expected lifetime utility in this steady state. By the results established in step 1, there exists $\tilde{b} = (\tilde{\rho}, \tilde{\mu}, 1, \tilde{\tau})$ with $\tilde{\tau} > 0$ such that $W(\tilde{b}) > W(b)$ for any $b = (\rho, \mu, 1, \tau)$ with $\tau \leq 0$.

Suppose $\bar{\theta}$ does not exists. Then we can find the sequences $\{b_n\} = \{(\rho_n, \mu_n, \theta_n, \tilde{\tau})\}$ and $\{b'_n\} = \{(\rho'_n, \mu'_n, \theta_n, \tau_n)\}$ with $\theta_n < 1$, $\tau_n \leq 0$ and $W(b_n) \leq W(b'_n)$ and with $(\theta_n, \tau_n) \to (1, \tau')$ for some $\tau' \leq 0$. Then we can find some compact $A \supset \{b_n, b'_n\}$ (refer to the proof of Proposition 1). By continuity of f and g and by compactness of A, I can show by the exact argument in the proof of Lemma 1 that T is compact-valued and u.h.c. on A. This argument also implies W is continuous on A.

By definitions of b_n and b'_n and T, $(\rho_n, \mu_n) \in T(b_n)$ and $(\rho'_n, \mu'_n) \in T(b'_n)$. Let $\hat{b} = (\hat{\rho}, \hat{\mu}, 1, \tilde{\tau})$ be a limit point of $\{b_n\}$, and let $\hat{b}' = (\hat{\rho}', \hat{\mu}', 1, \tau')$ be a limit point of $\{b'_n\}$. Because T is compact-valued and u.h.c., (by passing to subsequences) we have $(\hat{\rho}, \hat{\mu}) \in T(\hat{b})$ and $(\hat{\rho}', \hat{\mu}') \in T(\hat{b}')$. By the results established in step 1, $(\hat{\rho}, \hat{\mu})$ is the unique monetary steady state under $(1, \tilde{\tau})$, so $\hat{b} = \tilde{b}$. Also, $(\hat{\rho}', \hat{\mu}')$ is the unique monetary steady state under $(1, \tau')$. Now continuity of W implies $W(\tilde{b}) = W(\hat{b}) \leq W(\hat{b}')$, a contradiction. ■

Comparison with the Lagos-Wright model

First, I sketch a version of the LW model. At stage 1 of date t, people meet in a centralized market, where they can consume and produce. At stage 2 of date t, people meet in pairs in a decentralized market, where each person has an equal chance to be a buyer or a seller. The individual stage i utility from consuming q is $\tilde{u}_i(q)$ and disutility from producing q is $\tilde{c}_i(q)$. The discount factor between dates is $\beta \in (0, 1)$ (there is no within-date discount). The special assumption is $\tilde{c}_1(q) = q$. The initial money stock M_0 is evenly distributed among people at the start of date 0. Before the date t pairwise meetings, each person receives τM_t ($\tau > \beta - 1$) units of money transfer.

The centralized trade is competitive. The surplus from pairwise trade is split by generalized Nash bargaining in which the buyer's bargaining power is θ. A monetary equilibrium under (θ, τ) is a sequence $\{\rho_t\}$ such that when

$\rho_t > 0$ is the real balance in the date t centralized market, all people leave the market with M_t units of money; in particular, in equilibrium, the expected value for an agent holding zM_{t+1} $(z \leq 2)$ units of money at the end of date t is $z\beta\rho_{t+1} + \gamma$ for some constant γ.

For comparison, it is convenient to assume that in the OLG model, people can consume in their first stage with the utility function u_1 (this does not affect results in the main text). Let $(u_i, c_i) = (\tilde{u}_i, \tilde{c}_i)$, $i = 1, 2$, and $u_3 = \beta c_1$.

Let $\{\rho_t\}$ be a monetary equilibrium under (θ, τ) in the LW model. It is straightforward to show that $\{(\rho_t, \mu_d)\}$ ($supp\ \mu_d = \{1\}$) is a monetary equilibrium under (θ, τ) in the OLG model. (The key is that given $\{\rho_t\}$ the payoff for a young person holding zM_{t+1} units of money at the end of date t is $z\beta\rho_{t+1}$.) Evidently, the prices of money in the centralized market and the trade outcomes in the decentralized market in these two equilibria are identical. This is observational equivalence.

In the LW model, let $\tilde{\rho}(\tau)$ be the real balance in the unique monetary steady state under (θ, τ), and let $\tilde{V}(\tau)$ be the representative agent's steady-state expected discounted utility. Let $V(\tau)$ be the young person's steady-state expected lifetime in the OLG model. It is straightforward to show $V(\tau_2) - V(\tau_1) = (1-\beta)[\tilde{V}(\tau_2) - \tilde{V}(\tau_1)] + (1-\beta)[\tilde{\rho}(\tau_1) - \tilde{\rho}(\tau_2)]$. Now suppose in the LW model, M_0 is sold by the government to people in the date 0 centralized market (this does not affect the definition of equilibrium and observational equivalence). Then $V(\tau_2) - V(\tau_1) = (1-\beta)[\tilde{V}(\tau_2) - \tilde{V}(\tau_1)]$, so $\tilde{V}$ and V are maximized by the same rate of money transfer.

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