Welfare Theorems in a Market Game

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Abstract

This paper studies a multi-stage Shapley-Shubik market game with limit orders in a finite exchange economy. There is no discounting. The solution concept is subgame perfect equilibrium satisfying a weak refinement: an agent does not effectively exchange his resource with others in a subgame if the end-of-subgame allocation does not make him strictly better off than the start-of-subgame allocation. For a familiar class of preferences, the allocation of any equilibrium is efficient; and any efficient allocation strictly dominating the initial allocation or any Walrasian allocation for the initial allocation is an allocation of some equilibrium.

Keywords: Market game; Retrading; Market power; Walrasian equilibrium; Finite economy.

JEL classification: C72; D40; D51.

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1 Introduction

Walrasian equilibrium (WE) formalizes the idea that trading in a free market gives rise to efficiency by way of the price mechanism. Indeed, it is well known that under certain conditions, any Walrasian allocation (WA) is Pareto efficient and any efficient allocation is Walrasian with suitable redistribution of wealth. WE assumes price taking or perfect competition. But when individuals have market power, may trading *alone* be sufficient to attain efficiency in the free market? If efficiency happens to occur, would it be Walrasian with respect to the initial allocation and, what is the role played by the price mechanism? To address these issues, we study a version of Shapley and Shubik [20] (SS) market game with multi-round of trading (*retrading*) and with *limit orders* in a *finite* exchange economy, a strategic environment where prices are explicitly formed by actions of private agents and can be affected by each individual.

In our game, the trading posts keep open in the next round as long as some agent intends to trade at present. There is no discounting. If no agent intends to trade now, agents get a chance to submit orders sequentially in the next round and the game ends only if they choose no trading again. To ensure the feasibility of each agent's strategy independent of strategies of others, we follow Shapley and Shubik [20] to impose a *cash-in-advance* (CIA) constraint—all non-numeraire goods are only traded with the numeraire good, i.e., cash, and each agent's total biddings cannot exceed cash in hand when making bids. Our solution concept is subgame perfect equilibrium satisfying a weak *refinement*: an agent does not effectively exchange his resource with others in a subgame if the end-of-subgame allocation does not make him strictly better off than the start-of-subgame allocation. We establish two welfare theorems for a familiar class of preferences: (i) the allocation of any equilibrium is efficient; and (ii) any efficient allocation strictly dominating the initial allocation or any WA for the initial allocation is an allocation of some equilibrium.

The logic behind the first theorem is as follows. If the game is to end at an inefficient outcome, retrading gives agents opportunity to improve. No discounting eliminates exogenous trading costs to utilizing this opportunity. The sequential submission of orders before the purported ending allows an agent, a defector, to offer a trade for others to respond. Limit orders ensure that the trade, if carried out, is Pareto improving and, in particular, eliminate endogenous trading costs for the defector to initiate a departure from inefficiency. As such, sticking to inefficiency requires to compensate at least one non-defector. When the subgame following the initial Pareto-improving offer settles on inefficiency, the defector can keep on deviating. With the aforementioned refinement, at least for one agent, when he becomes the defector, the limit allocation resulting from his deviation must be inefficient, implying that at some point there is no sufficient room to compensate non-defectors. In short, costless retrading drives out inefficiency.

For the second theorem, we first show that when moving to a WA for the initial allocation, agents can all rely on anonymous information and they do behave as if there were price taking on the equilibrium path. Specifically, we show that agents can trade to the WA by using the associated WP as the reservation price; here, in particular, costless retrading resolves the *liquidity problem* caused by the CIA constraint. But non-anonymous information is needed for off-path coordination when a deviation of an agent to a different reservation price can potentially lead to *advantageous redistribution*. Specifically, we show that following such a deviation, agents can trade to an allocation in each agent's budget frontier defined by another WP associated with another efficient allocation that is chose to deter the deviation from occurring; here costless retrading redistributes among agents wealth defined by the new WP. Such wealth redistribution also allows agents to reach efficient allocations that are not WAs for the initial allocation but strictly dominate the initial allocation.

In static SS games, the individual market power can be confined by Bertrand competition under limit orders as in Dubey [1] or by short sale as in Peck and Shell [15] so that all active Nash equilibria are Walrasian. Dubey [1] and Peck and Shell [15] use money or credit to resolve the liquidity problem, calling for outside enforcement or equivalently, the strategic budget constraint, to prevent default on off-equilibrium paths. So, in contrast to our game, those static games rely on certain no-trading arrangement to achieve efficiency.[1] Dubey et al. [4] introduce retrading into SS games but focus on a large economy, where retrading does force all active Nash equilibria to approach WE. For a finite economy, Ghosal and Morelli [11] find that retrading can lead some subgame perfect equilibrium to approach WE but, as their game does not admit limit orders, inefficiency arises in other equilibria.

[1]This applies to other static SS games using fiat money or credit to obtain sorts of coincidence of WAs and Nash equilibrium allocations; see, e.g., Dubey and Shubik [3], Postlewaite and Schmeidler [16], Dubey and Geanakoplos [2], and Dubey and Shapley [5].

Rubinstein and Wolinsky [18] study a finite number of buyers and sellers who exchange an indivisible good with transferable utility through sequential bilateral matching and bargaining; they find many sequential equilibria with non-Walrasian prices and, when agents have heterogeneous valuations, many inefficient equilibria.[2] In our game, inefficient equilibria can arise without the aforementioned refinement; analogies of this refinement, however, do not seem to be effective in the Rubinstein-Wolinsky [18] game.

2 Economy and the market game

There are $I \geq 2$ agents and $L \geq 2$ types of endowed goods. Denoting by $u_i(c_i)$ agent i's utility from consuming $c_i = (c_{i1}, ..., c_{iL}) \in \mathbb{R}_+^L$ (c_{il} is the consumption of good l), we maintain the following assumption on the utility functions throughout.

Assumption 1 *For each $1 \leq i \leq I$, u_i is strictly increasing in each of L arguments and twice differentiable.*

Given I, L, and the utility functions $(u_1, ...u_I)$, we identify an *economy* with a vector of *aggregate endowments* $W = (W_1, ...W_L) \in \mathbb{R}_{++}^L$ (W_l is the aggregate endowment of good l). For an economy W, a feasible *allocation* is a vector $\omega = (\omega_1, ..., \omega_I) \in \mathbb{R}_+^{I \times L}$, where $\omega_i = (\omega_{i1}, ..., \omega_{iL}) \in \mathbb{R}_+^L$ is agent i's *endowment* and $\sum_i \omega_{il} = W_l$; Ω is the set of all feasible allocations. We denote by $(c(\omega), p(\omega))$ a Walrasian equilibrium (WE) for $\omega \in \Omega$, where $c(\omega) \in \Omega$ is a *Walrasian allocation* (WA) and $p(\omega)$ is a *Walrasian price* (WP) for ω. The *initial allocation*, denoted ω^0, has $\omega_i^0 \neq 0 \in \mathbb{R}_+^L$ all i.

The market game is a multiple-round dynamic game. A round of the game is a trading round, a test-for-ending round, or a terminal round. The game starts with a trading round. In a *trading round*, there are $L-1$ trading posts opening: good 1 is the numeraire or cash and good $l \geq 2$ is traded for cash at trading post l. In trading post l, agent i submits a *limit order* (s_{il}, b_{il}, p_{il}) with $p_{il}, s_{il}, b_{il} \geq 0$ and $s_{il}b_{il} = 0$. When

[2] The matching-bargaining framework is developed by Rubinstein and Wolinsky [17]; see Gale [9] for a survey. Using this framework, Gale [7, 8] considers a finite number of divisible goods (as us) but focuses on a large economy. Gale [7] shows that every perfect equilibrium outcome is Walrasian; McLennan and Sonnenschein [14] obtain the same result in a related setting. With homogeneous valuations, the only equilibrium that presents the Walrasian price in the Rubinstein-Wolinsky [18] game is the one using anonymous information. Sabourian [19] shows that aversion to complexity of strategies can play the same role as anonymous information; Gale and Sabourian [10] extend the result to heterogeneous valuations.

$s_{il} > 0$, the order states that i is willing to sell any amount of good l in the interval $[0, s_{il}]$ at the reservation price p_{il} or higher, and 0 at any lower price; when $b_{il} > 0$, the order states that i is willing to buy good l by any amount of cash in the interval $[0, b_{il}]$ at the reservation price p_{il} or lower, and 0 at any higher price. An order with $(s_{il} + b_{il})p_{il} = 0$ is an *inactive order*, meaning that the agent does not buy or sell any amount of good l; an order with $(s_{il} + b_{il})p_{il} > 0$ is an *active order*. If the price at post l turns out to be p_{il}, the quantity of good l sold (spending of good 1, resp.) by agent i is determined by the standard rationing scheme, which proportionally scales down the quantities (spendings, resp.) of all agents who quote the price equal to p_{il} to meet the aggregate demand (supply, resp.) at the price p_{il}. The set of i's *trading actions* (i.e., orders to submit) can be represented by

$$\Gamma_i(\omega_i) = \{\gamma_i = (s_i, b_i, p_i) : s_{il} \leq \omega_{il},\ s_{il}b_{il} = 0,\ \sum_{l\geq 2} b_{il} \leq \omega_{i1}\}, \tag{1}$$

where ω_i is i's pre-trading endowment, $s_i = (s_{i2}, ..., s_{iL})$, $b_i = (b_{i2}, ..., b_{iL})$, $p_i = (1, p_{i2}, ..., p_{iL})$, and $(s_{il}, b_{il}, p_{il}) \in \mathbb{R}^3_+$ all $l \geq 2$; the constraint $\sum_{l\geq 2} b_{il} \leq \omega_{i1}$ is the cash-in-advance (CIA) constraint. The trading at post l is determined by a market-clearing price (if there is any); the allocation resulting from trading in all posts becomes the allocation at the start of the next round.

Lemma 1 *If ω is the allocation prior to trading at the present round and agents take a profile of trading actions $(\gamma_1, ..., \gamma_I) \equiv \gamma \in \Gamma(\omega) \equiv \times_{1\leq i\leq I}\Gamma_i(\omega_i)$, then there is a well-defined market-clearing price $p_l(\gamma)$ for each good $l \geq 2$ and the trading gives rise to a unique allocation, denoted $\lambda(\omega, \gamma)$.*

Proof. See the appendix. ■

The game reaches a *test-for-ending round* if the previous round is a trading round and all agents submit inactive orders at all trading posts in the previous round. A test-for-ending round is a special sort of trading round where Lemma 1 applies. In such a round, each trading post has *two stages* to accept orders: stage 1 only accepts active orders; stage 2 accepts both active and inactive orders. Each agent can choose either stage to submit an active order and all his orders must satisfy (1). If at least one agent submits at least one active order in this round, then the game proceeds to a trading round; otherwise, the game reaches a terminal round. When a *terminal* *round* is reached, agents consume the allocation in hand and that allocation is an *outcome* of the game.

The initial allocation ω^0 and the utility functions are public information. All orders become public information right after being submitted. This completes the description of our market game.

Three remarks of our game are in order. First, the form of limit orders in our game is studied by Liao [13] and differs from the form in Dubey [1], Simon [21], and Dubey et al. [4]; we adopt this form because it implies those nice properties in Lemma 1. Secondly, sequential submission of orders in test-for-ending rounds eliminates no trading as a self-fulfilling event (one submits inactive orders simply because he anticipates other agents to do so); its substantial usage is discussed below. We can let each trading round have sequential submission of orders; the current setup takes a minimal departure from the conventional setup and simplifies the description of equilibrium strategies in section 4. Lastly, some subgames in our game do not have terminal rounds and, hence, violate the free-participation constraint as no one has a chance to consume. We may alternatively grant each agent an option to exit the game at the end of a trading round;[3] for expositional consideration, we let free participation be part of the equilibrium condition (condition (a) in Definition 1 below).

A (pure) *strategy* of agent i is denoted by $f_i = \{f_{it}\}_{t \geq 1}$. Provided that ω_i is the endowment of i at the start of round t, when round t is a trading round, f_{it} assigns to each history up to the start of round t a trading action $\gamma_i \in \Gamma_i(\omega_i)$; when round t is a test-for-ending round, f_{it} assigns to each history up to the start of stage 1 of round t some subset K of $\{2, ..., L\}$ and active orders $\{(s_{il}, b_{il}, p_{il}) : l \in K\}$ and to each history up to the start of stage 2 $\{(s_{il}, b_{il}, p_{il}) : l \notin K\}$ such that all orders constitute a trading action of $\Gamma_i(\omega_i)$.

Definition 1 *A profile of strategies $f = (f_1, f_2, ..., f_I)$ is an equilibrium of the market game if it is a subgame perfect equilibrium of the game and if when it specifies the outcome ω' for a subgame that starts with ω, (a) $u_i(\omega'_i) \geq u_i(\omega_i)$ all i and (b) $\omega'_i = \omega_i$ for any i with $u_i(\omega'_i) = u_i(\omega_i)$.*

[3] To eliminate the situation that one exits simply because he anticipates others to do so, we may assume that an agent can take the exit option at the end of a round only if his orders are all inactive in this round and the last round. Then, the only substantial issue is how to define efficiency when agents who do not exit do not hold any good 1. We may let availability of liquidity be part of the condition for efficiency or allow remaining agents to use another good as the numeraire. Either way, our argument can go through with straightforward adaptation.

In Definition 1, condition (a) represents the free-participation constraint. Condition (b) says that an agent does not effectively exchange his resource with others when he sees not strict benefit from exchanging in any subgame; the use of this refinement is discussed below.

Let $u_{il}(c_i)$ denote the marginal utility of consumption of good l for agent i evaluated at c_i. Our main results are the following.

Theorem 1 *Suppose u_i is strictly quasiconcave all i and $\lim_{c_{il}\downarrow 0} u_{il}(c_i) = \infty$ all (i, l). Then for any economy W outside a discrete set of $\mathbb{R}^L_{++}$, the outcome of the market game specified by any equilibrium is efficient.*

Theorem 2 *Suppose u_i is strictly concave. Then for any economy $W \in \mathbb{R}^L_{++}$, any efficient allocation ϕ^0 is the outcome of specified by some equilibrium of the market game when ϕ^0 strictly dominates ω^0 or when $\phi^0 = c(\omega^0)$.*

Theorems 1 and 2 are proved in sections 3 and 4, respectively.

3 Proof of Theorem 1

We begin with an outline of our proof. Let f be an equilibrium and suppose that f specifies an inefficient allocation ω as the outcome of some subgame. Because the marginal utility of increasing consumption of cash from zero is infinite, there exists a limit order for each agent such that if the order is carried out in the trading post, the trade benefits the agent and some other agent. This is the content of Lemma 2. Example 2 provides an example that Theorem 1 fails when the marginal utilities of consuming cash are finite.

The deviation in concern is that an arbitrary agent, the defector, submits the aforementioned order at stage 1 in the test-for-ending round right before the relevant terminal round. Because f is an equilibrium, no other agent should respond to the order at stage 2. This non-responding implies that the outcome ω' of the continuation subgame strictly dominates ω for some other agent. For the defector, ω' is indifferent to ω; thus, by condition (b) in Definition 1, other agents can only rearrange their endowments in ω to reach ω'. If ω' is inefficient, the defector can keep on deviating by the same manner and the sequence of deviation must lead to some limit allocation.

What is critical is that at least for one agent, the limit allocation is inefficient. The result is established by Lemma 3 and Lemma 4 for W outside a discrete set of

$\mathbb{R}^L_{++}$. Recall that Theorem 1 assumes $\lim_{c_{il}\downarrow 0} u_{il}(c_i) = \infty$ all (i, l), implying that all efficient allocations in concern are interior. This implication is used in the proof of Lemma 3 for a property of an efficient limit allocation. Theorem 1 also assumes that u_i is strict quasiconcave, implying that the optimal demand of agent i for good l in the Walrasian market is single-valued. This implication is used in the proof of Lemma 4.

Now focus on an agent whose limit allocation is inefficient. When this agent becomes the defector, the inefficient limit approximates the start-of-subgame allocation and the end-of-subgame allocation of some subgame (after he starts to defect). Inefficiency of the limit implies a positive lower bound on the benefit for another agent, say agent j, to trade with the defector after the defector submits the order at the start of the subgame; this lower bound exceeds the benefit that all agents other than the defector can give to j in the subgame because the start-of-subgame allocation and the end-of-subgame allocation are sufficiently close.

Below we first show by examples relevance of some setups in our game, equilibrium condition, and assumption to carry out this outline. We next establish intermediate results (Lemmas 2-4) and then complete the proof.

Throughout, let $U_{il}(c_i) = u_{il}(c_i)/u_{i1}(c_i)$ (the marginal substitution rate between good l and good 1 of agent i at c_i).

3.1 Examples

The following example illustrates roles played by test-for-ending rounds, limit orders, and condition (b) in Definition 1.

Example 1 $I = 2$, $L = 2$, $u_i(c_i) = (c_{i1}c_{i2})^{1/2}$, and $W_1 = W_2 = 2$.

For Example 1, $p(\omega) = (1, 1)$ all $\omega \in \Omega$. Suppose the game ends at $\omega = ((1.6, 0.4), (0.4, 1.6))$; note $c(\omega) = ((1, 1), (1, 1))$. Without loss of generality, let agent 1 be the potential defector who makes an active order at stage 1 of the test-for-ending round right before the terminal round. First, such a deviation is impossible in the absence of test-for-ending rounds. Secondly, suppose reservation prices cannot be used. Without loss of generality, suppose the order of agent 1 is to bid $b > 0$ amount of cash to buy good 2. Then agent 2 can respond to offer an arbitrarily small amount of good 2 in stage 2. Agent 1 is worse off as the price of good 2 can be arbitrarily large.

Lastly, suppose condition (b) in Definition 1 is not imposed. Although the order of agent 1 can strictly benefit both agents (e.g, the order is to spend up to 0.6 amount of cash under the reservation price 1), the equilibrium can be arranged so that agent 2 can gain more by not responding to the order. Indeed, consider the subgame following that agent 2 submits an inactive order at stage 2 and let the equilibrium in the subgame be the one that gives all trading surplus to agent 2 (in any trading round, agent 2 always submits an order to sell up to 0.4 units of good 2 under the reservation price 2; agent 1 submits an order to buy good 2 under the reservation price 2 by spending up to 0.8 units of cash). The subgame ends at ((0.8,0.8), (1.2,1.2)), which is sufficient to deter the deviation of agent 1.

Next we use a different example to illustrate the role played by the infinite marginal utility of increasing consumption of cash from zero.

Example 2 $I = 2$, $L = 3$, $u_1(c_1) = c_{11} + (c_{12}c_{13})^{1/2}$, $u_2(c_2) = 0.1c_{21} + (c_{22}c_{23})^{1/2}$, and $W_1 = W_2 = W_3 = 2$.

For Example 2, efficient allocations satisfy $c_{11} = 2$, $c_{12} = c_{13}$, and $c_{22} = c_{23}$. Suppose the game ends at $\omega = ((2, 1.6, 0.4), (0, 0.4, 1.6))$. In order to depart from ω, agent 2 must first sell some good 2 or 3 to agent 1 to acquire cash for any further trading. But because $U_{12}(\omega_1) = 0.25$, $U_{13}(\omega_1) = 1$, $U_{22}(\omega_2) = 10$, and $U_{23}(\omega_2) = 2.5$, any allocation ω' resulting from the selling of agent 2 must make one agent, say i, strictly worse off than ω. Then in the subgame starting at ω', let the equilibrium be the one that gives all trading surplus to agent -i, which is sufficient to prevent the deviation from occurring.

3.2 Intermediate results

Following the outline given above, we first construct an order for each agent which can be used by the agent to initiate a departure from an inefficient allocation.

Lemma 2 *Let ω be inefficient and satisfy $\omega_i \neq 0$ all i. For each i, there exists an order $\zeta^i(\omega)$ and some $j \neq i$ such that agents i and j strictly benefit (with respect to ω_i and ω_j) from carrying out the order.*

Proof. Fix i and consider three cases; in each case, let $p_l = 0.5U_{jl}(\omega_j) + 0.5U_{il}(\omega_i)$ and let $\epsilon > 0$ be sufficiently small. The first case is that all agents have cash. Then for some $l \geq 2$, $U_{il}(\omega_i) \neq U_{jl}(\omega_j)$ for some j. If $U_{il}(\omega_i) < U_{jl}(\omega_j)$, then $\zeta^i(\omega) = (\epsilon, 0, p_l)$ (i sells good l); otherwise, $\zeta^i(\omega) = (0, \epsilon, p_l)$ (i buys good l). The second case is that agent i does not have cash. Then $\omega_{il} > 0$ for some $l \geq 2$, $U_{il}(\omega_i) = 0$, and there is some j such that $\omega_{j1} > 0$ and $U_{jl}(\omega_j) > 0$. Now $\zeta^i(\omega) = (\epsilon, 0, p_l)$ (i sells good l). The third case is that agent i has cash but some agent j does not. Then $\omega_{jl} > 0$ for some $l \geq 2$ and $U_{jl}(\omega_j) = 0$. Now $\zeta^i(\omega) = (0, \epsilon, p_l)$ (i buys good l). ■

Let f be an equilibrium which specifies an inefficient outcome ω for a subgame. For each $1 \leq i \leq I$, we define the sequence of allocations $\{\omega_n^i\}_{n=1}^{\infty}$ with $\omega_1^i = \omega$ by induction as follows. When ω_n^i is inefficient, denote by $G^i(\omega_n^i)$ the subgame following that agent i submits the order $\zeta^i(\omega_n^i)$ at stage 1 of the test-for-ending round right before the terminal round that ends the game with ω_n^i, and denote by $g^i(\omega_n^i)$ the outcome of $G^i(\omega_n^i)$ specified by f; let $\omega_{n+1}^i = g^i(\omega_n^i)$. When ω_n^i is efficient, let $\omega_{n+1}^i = \omega_n^i$.

Lemma 3 *Suppose the outcome ω of a subgame specified by an equilibrium f is inefficient. Then for each $1 \leq i \leq I$ the sequence of allocations $\{\omega_n^i\}$ converges to some allocation ω^i and $\omega_{ni}^i = \omega_i^i$ all n. Moreover, if ω^i is efficient, then $p^i(\omega)\omega_k^i = p^i(\omega)\omega_k$ all $k \neq i$, where $p^i(\omega)$= $(1, U_{i2}(\omega_i), ..., U_{iL}(\omega_i))$.*

Proof. Fix i. If $g^i(\omega_n^i)$ is efficient for some n, then $\omega_{n'}^i = g^i(\omega_n^i)$ all $n' \geq 1$. So it suffices to consider the case that for each n, $g^i(\omega_n^i)$ is inefficient. Because f is an equilibrium, $u_k(\omega_{n+1k}^i) \geq u_k(\omega_{nk}^i)$ all $1 \leq k \leq I$. It follows that the sequence $\{(u_1(\omega_{n1}^i), ..., u_I(\omega_{nI}^i))\}_n$ of the utility vectors is non-decreasing in the pointwise sense. Because the aggregate endowments are finite, the sequence of the utility vectors must converge, implying that the sequence $\{\omega_n^i\}$ of allocations converges. Because f is an equilibrium, $u_i(\omega_{n+1i}^i) = u_i(\omega_{ni}^i)$, implying $\omega_{n+1i}^i = \omega_{ni}^i$ (condition (b) in Definition 1). It follows that $\omega_{ni}^i = \omega_i^i$ all n.

Now suppose that ω^i is efficient. In the subgame $G^i(\omega_n^i)$, agents other than i must actively trade among themselves because $\omega_{n+1j}^i \neq \omega_{nj}^i$ for some $j \neq i$. The market-clearing price p_l at trading post l for any such trade must be equal to $U_{il}(\omega_i)$. For, otherwise, agent i can gain by either selling or buying a sufficiently small amount of good l; recall that $\lim_{c_{im} \downarrow 0} u_{im}(c_i) = \infty$ all m, implying $\omega_{im} = \omega_{im}^i > 0$ (i.e., agent i has cash to buy and has good l to sell). Because the net trade of each agent has the zero value under the price vector $(1, p_2, ..., p_L)$, it follows that $p^i(\omega)\omega_{nk}^i = p^i(\omega)\omega_{n+1k}^i$ all $k \neq i$ for each n and, hence, that $p^i(\omega)\omega_k = p^i(\omega)\omega_k^i$ all $k \neq i$. ■

To obtain an inefficient Lemma-3 limit ω^i for some i (given ω is inefficient), suppose by contradiction that ω^i is efficient for each $1 \leq i \leq I$. Then by Lemma 3, $p^i(\omega)$ is a WP for ω for each i. This is obviously impossible for any economy W when $I = 2$ (actually it is impossible for any W when $I = 3$); for a general I, however, we have to restrict W a little bit.

Lemma 4 *For W outside a discrete set of $\mathbb{R}^L_{++}$, if the outcome ω of a subgame specified by an equilibrium f is inefficient, then the Lemma-3 limit ω^i is inefficient for some i.*

Proof. It suffices to show that for W outside a discrete set of $\mathbb{R}^L_{++}$, if $x \in \Omega$ is inefficient, then it is impossible that $p^i(x)$ is a WP for x all i. To this end, let $x_i = (x_{i1}, ..., x_{iL}) \in \mathbb{R}^L_{++}$, $x = (x_1, ..., x_I)$, $w = (w_1, ..., w_I) \in \mathbb{R}^L_{++}$, $\vartheta_i = (\vartheta_{i2}, ..., \vartheta_{iI}) \in \mathbb{R}^{L-1}_{++}$, and $\vartheta = (\vartheta_1, ..., \vartheta_I)$. Let $q_{jl}(\vartheta_i, x_j\vartheta_i)$ denote the optimal demand of agent j for good l in the Walrasian market when the price vector is $(1, \vartheta_i)$ and his endowment is x_j. Let

$$\Phi^x_{il}(x, \vartheta, w) = \sum_{j \neq i} q_{jl}(\vartheta_i, x_j\vartheta_i) - w_l + x_{il}$$

all (i, l); let

$$\Phi^\vartheta_{il}(x, \vartheta, w) = \vartheta_{il} - U_{il}(x_i)$$

all i and $l \geq 2$; and let

$$\Phi^w_l(x, \vartheta, w) = \sum_i x_{il} - w_l$$

all l. Now $(x, \vartheta, w) \mapsto \Phi(x, \vartheta, w)$ is a mapping from $\mathbb{R}^{I \times L}_{++} \times \mathbb{R}^{I \times (L-1)}_{++} \times \mathbb{R}^L_{++}$ to $\mathbb{R}^{I \times L} \times \mathbb{R}^{I \times (L-1)} \times \mathbb{R}^L$, where $\Phi = (\Phi^x_{11}, ..., \Phi^x_{IL}, \Phi^\vartheta_{12}, ..., \Phi^\vartheta_{IL}, \Phi^w_1, ..., \Phi^w_L)$. Observe that $\Phi(x, \vartheta, w) = 0$ iff $p^i(x)$ is a WP for the allocation x all i when w is the aggregate endowments. Let $F = \{(x, \vartheta, w) : \Phi(x, \vartheta, w) = 0 \text{ and } x \text{ is inefficient}\}$. It suffices to show that the Jacobian of Φ evaluated at any point of F is invertible. For if so then by the preimage theorem, F is a zero-dimension manifold. The check of that Jacobian is delegated to the appendix. ■

3.3 Completion of proof

Let f be an equilibrium and suppose by contradiction that the outcome ω of a subgame specified by f is inefficient. Suppose without loss of generality that for agent

1, his Lemma-3 limit allocation ω^1 is inefficient. Referring to Lemma 2, let ρ be the allocation after the order $\zeta^1(\omega^1)$ is carried out between agents 1 and j. By Lemma 2, $\Delta_j \equiv u_j(\rho_j) - u_j(\omega_j^1) > 0$ and $\Delta_1 \equiv u_1(\rho_1) - u_j(\omega_1^1) > 0$. Let ρ_n be the allocation after the order $\zeta^1(\omega^1)$ is carried out between agents 1 and j when the allocation is ω_n^1. When n is sufficiently large, $u_j(\rho_{nj}) > u_j(\omega_j^1) + 0.5\Delta_j$ (ρ_{nj} is sufficiently close to ρ_j) and $u_1(\rho_{n1}) > u_1(\omega_1^1) + 0.5\Delta_1$ (ρ_{n1} is close to ρ_1); also, $u_j(\omega_{n+1j}^1) - 0.25\Delta_j < u_j(\omega_{nj}^1) < u_j(\omega_j^1) + 0.25\Delta_j$ (ω_{nj}^1 is close to ω_j^1 and ω_{n+1j}^1). Because $u_1(\omega_{n1}^1) = u_1(\omega_1^1)$ all n, if agent 1 submits $\zeta^1(\omega^1)$ at stage 1 of the test-for-ending round right before the terminal round that ends the subgame $G^1(\omega_{n-1}^1)$ at $\omega_n^1 = g^1(\omega_{n-1}^1)$ when n is large, agent j must respond at stage 2 ($u_j(\rho_{nj}) > u_j(\omega_{n+1j}^1)$) and, hence agent 1 is better off by his deviation ($u_1(\rho_{n1}) > u_1(\omega_1^1) = u_1(\omega_1)$), a contradiction.

4 Proof of Theorem 2

To construct the equilibrium f in Theorem 2, we proceed by four steps. At step 1, we provide a solution to the liquidity problem due to the CIA constraint. At step 2, we show why limit orders may help contain the individual market power in applying the step-1 solution and why in general containing the individual market power may involve some wealth redistribution. At step 3, we establish a suitable wealth-redistribution path. Theorem 2 assumes strict concavity of u_i, which plays a role in constructing the step-3 wealth-redistribution path (including allowing us to apply the second fundamental theorem of welfare economics). Using the intermediate results obtained from steps 1-3, we complete the proof at step 4.

4.1 A solution to the liquidity problem

Given the CIA constraint, some agents may not have sufficient amount of cash even when all agents intend to reach an efficient allocation. This liquidity problem can be solved by retrading. To begin with, let $X(z, p) = \{x \in \Omega : px_i = pz_i \text{ all } i\}$ for any $z \in \Omega$ and any price vector $p \in \mathbb{R}_{++}^L$ with $p_1 = 1$. A key fact about this subset of Ω is that any x in it has a neighborhood $B(x)$ such that if agents start with $x' \in B(x)$ and use the common reservation price p, then they can reach any $x'' \in B(x)$ by two rounds of trading. As implied by this fact, agents can reach any $x' \in X(z, p)$ from any $x \in X(z,p)$ after R rounds of trading for some fixed R.

Formally, we say that a profile of trading actions $\gamma \in \Gamma(z)$ (see Lemma 1) is a *p-trading profile* for a price vector p if p_i in γ_i is equal to p all i. Agents can reach $z' \in \Omega$ from $z \in \Omega$ (or move from z to z') by one round of p-trading if $z' = \lambda(z,\gamma)$ (see Lemma 1) and γ is a p-trading profile. Agents can reach z' from z by R rounds of p-trading for a common p if there exists a sequence of allocations $\{x(r)\}_{r=0}^{R}$ such that $x(0) = z$, $x(R) = z'$, and agents can reach $x(r+1)$ from $x(r)$ by one round of p-trading for $0 \leq r \leq R-1$. A subset B of $X(z,p)$ is *linked by R rounds of p-trading* if for any $(x,x') \in B \times B$ agents can reach x' from x by R rounds of p-trading.

Lemma 5 *Let $X(z,p)$ be equipped with the relative topology induced by the Euclidean topology of $\mathbb{R}^{I \times L}$. Then any x in the set $X(z,p)$ has an open neighborhood $B(x)$ that is linked by 2 rounds of p-trading.*

Proof. Fix $x \in X(z,p)$. Pick i such that $x_{i1} > 0.5W_1/I$. Let $d_j(x',x'') = \sum\limits_{l\geq 2} \max\{p_l(x'_{jl} - x''_{jl}), 0\}$, which is cash acquired by agent j if he starts with x'_j and reaches x''_j by one round of p-trading. Let $d(x',x'') = \sum\limits_{j\neq i} d_j(x',x'')$. Let $B(x)$ be a neighborhood of x in $X(z,p)$ such that if $x' \in B(x)$ then $x'_{i1} > 0.25W_1/I$ and that if $x',x'' \in B(x)$ then $d(x',x'') < 0.25W_1/I$. Now fix $x',x'' \in B(x)$ and consider the following two rounds of p-trading.

Round 1. Each agent $j \neq i$ sells $x'_{jl} - x''_{jl}$ units of good $l \geq 2$ when $x'_{jl} > x''_{jl}$; agent i buys all goods sold by the other agents. Formally, the trading-action profile γ has $(s_{il}, b_{il}, p_{il}) = (0, \sum\limits_{j\neq i} \max\{x'_{jl} - x''_{jl}, 0\}p_l, p_l)$ and $(s_{jl}, b_{jl}, p_{jl}) = (\max\{x'_{jl} - x''_{jl}, 0\}, 0, p_l)$ if $j \neq i$ for each $l \geq 2$.

Round 2. Each agent $j \neq i$ buys $x''_{jl} - x'_{jl}$ units of good $l \geq 2$ when $x''_{jl} > x'_{jl}$; agent i supplies the exact amount. Formally, the trading-action profile γ has $(s_{il}, b_{il}, p_{il}) = (\sum\limits_{j\neq i} \max\{x''_{jl} - x'_{jl}, 0\}, 0, p_l)$ and $(s_{jl}, b_{jl}, p_{jl}) = (0, \max\{x''_{jl} - x'_{jl}, 0\}p_l, p_l)$ if $j \neq i$ for each $l \geq 2$. After trading, each $j \neq i$ reaches x''_j so i reaches x''_i too. ■

Lemma 5 uses the observation that locally at least one agent (agent i in the above proof) is not CIA constrained and his trades provide liquidity to other agents. This local solution to the liquidity problem can be extended globally.

Lemma 6 *The set $X(z,p)$ is linked by finite rounds of p-trading.*

Proof. Let $B(x)$ be a 2-round linked open neighborhood of $x \in X(z,p)$. The collection of sets $\{B(x) : x \in X(z,p)\}$ constitutes an open cover for $X(z,p)$. Because $X(z,p)$ is compact, the open cover has a finite cover; that is, there exist some $y(1), ..., y(N) \in X(z,p)$ such that $\bigcup_{1 \le n \le N} B(y(n)) = X(z,p)$. Fix $x, x' \in X(z,p)$ and, without loss of generality, suppose $x \in B(y(1))$ and $x' \in B(y(N))$. Because X is connected (it is convex), agents can reach x' from x by no more than $2N$ rounds of p-trading. ∎

Our application of Lemma 6 pertains to the scenario that agents move from z to $c(z)$ by p-trading with $p = p(z)$ (recall that $(c(z), p(z))$ is a WE for z); for future reference, it is convenient to fix a p-trading profile when agents start moving.

Definition 2 *Given $z \in \Omega$ and $(c(z), p(z))$, pick minimal rounds of p-trading with $p = p(z)$ by which agents can move from z to $c(z)$, and let $a(z) = (a_1(z), ..., a_I(z))$ denote the first-round profile.*

4.2 The individual market power

What may incentivize agent i to take the trading action $a_i(z)$ in Definition 2? The following implication of limit orders provides part of the answer.

Lemma 7 *Fix a price vector $p \in R^L_{++}$ with $p_1 = 1$. Fix i and let $\gamma \in \Gamma(z)$ satisfy $p_j = p$ in γ_j for each agent $j \neq i$. Then $p\lambda_i(z,\gamma) \le pz_i$.*

Proof. Let $p_l(\gamma)$ be the market-clearing price at trading post l. We claim that $p_l[\lambda_{il}(z,\gamma) - z_{il}] \le p_l(\gamma)[\lambda_{il}(z,\gamma) - z_{il}]$ for each $l \ge 2$, implying $p[\lambda_{il}(z,\gamma) - z_i] \le p(\gamma)[\lambda_{il}(z,\gamma) - z_i]$. Because the net trade $\lambda_{il}(z,\gamma) - z_i$ of agent i has the zero value under the price vector $p(\gamma)$, we have $p\lambda_i(z,\gamma) \le pz_i$. To see the claim, notice that either $\lambda_{il}(z,\gamma) > z_{il}$ or not. If the former (i.e., agent i is a buyer of good l), then agent i cannot purchase good l at a price strictly lower than p_l so $p_l(\gamma) \ge p_l$ and the claim holds. If the latter (i.e., agent i is a seller of good l), then agent i cannot sell good l at a price strictly higher than p_l so $p_l(\gamma) \le p_l$ and the claim holds too. ∎

By Lemma 7, one agent cannot increase his wealth measured by p if all other agents use p as the reservation price. The following is an example that p-trading itself is sufficient to incentivize agents to reach $c(z)$ from z.

Example 3 $I = 3$, $L = 3$, $u_i(c_i) = (c_{i1}c_{i2}c_{i3})^{1/2}$, and $W_1 = W_2 = W_3 = 3$.

For Example 3, $c(\omega^0)$ is the outcome of the game specified by some equilibrium f for any ω^0. To see this, note that for any $z \in \Omega$, $p(z) = p = (1, 1, 1)$. Let f be such that if a trading round starts with z, then agent i's trading action is $a_i(z)$ in Definition 2. Suppose agent i deviates and the resulting allocation is z'. Because $pz'_i \leq pz_i$, i cannot benefit when $c(z')$ is the equilibrium outcome of the continuation subgame.

The above strategy profile encounters a problem when a deviation of agent i results in $p(z)z'_i < p(z)z_i$ while $p(z') \neq p(z)$ and $u_i(c_i(z')) > u(c_i(z))$, i.e., when the deviation leads to advantageous redistribution favoring i (see, Gale [6] and Guesnerie and Laffont [12]). To deter such a deviation, it is necessary to let the equilibrium outcome in the subgame following the deviation be some efficient z'' such that $z' \notin X(z'', p(z''))$ and $u_i(c_i(z)) > u_i(z''_i)$.

Lemma 8 *Suppose $z' \notin X(z, p(z))$ is not efficient and $u_i(c_i(z)) > u_i(z'_i)$ for some i. There exists $z'' \in \Omega$ such that z'' is efficient, $u_j(z''_j) > u_j(z'_j)$ all $j \neq i$ and $u_i(c_i(z)) > u_i(z''_i) > u_i(z'_i)$.*

Proof. See the appendix. ■

Condition 1 *$u_i(\phi_i) > u_i(\omega_i)$ all i; $\phi \in \Omega$ is efficient; and $\phi \neq c(\omega)$.*

Condition 1 is satisfied when (ϕ, ω) is (z'', z') in Lemma 8. Also, Condition 1 is satisfied when (ω, ϕ) is (ω^0, ϕ^0) in Theorem 2 in case $\phi^0 \neq c(\omega^0)$ strictly dominates ω^0. So containing the individual market power points to the same wealth-redistribution problem as that case in Theorem 2.

4.3 A wealth-redistribution path

For the above wealth-redistribution problem, consider (ϕ, ω) satisfying Condition 1. Fix $x \notin X(\phi, p(\phi))$ with $u_i(\phi_i) > u_i(x_i)$ all i (x may or may not be equal to ω). Decompose $A \equiv \{1, ..., I\}$ to

$$A^+(x, \phi) = \{k : p(\phi)x_k > p(\phi)\phi_k\} \text{ and } A^-(x, \phi) = \{j : p(\phi)x_j < p(\phi)\phi_j\}. \quad (2)$$

If agents start with x and end up with consuming ϕ, then agents in $A^+(x, \phi)$ must lose some of their wealth evaluated at $p(\phi)$ to agents in $A^-(x, \phi)$. Let $\{x^-_{jk} : (k, j) \in$

$A^+(x,\phi) \times A^-(x,\phi)\}$ be a set of transfers of goods resulting from this wealth redistribution, where $x^-_{jk} \in \mathbb{R}^L_+$ stands for the transfer of goods from agent k to agent j. The transfer should obey the rule of the game—k cannot simply hand in x^-_{jk} to j.

To prevent agents from exploiting the relevant trading process, we first find a (finite) sequence in Ω that leads x to an allocation in $X(\phi, p(\phi))$ such that each allocation in the sequence is strictly dominated by ϕ. For this sequence, let us start with a part x^+_k of the endowment x_k for agent $k \in A^+(x,\phi)$ satisfying

$$p(\phi)x^+_k = p(\phi)(x_k - \phi_k). \tag{3}$$

If k keeps $x_k - x^+_k$ and transfers all x^+_k to agents in $A^-(x,\phi)$, then

$$\textstyle\sum_{j\in A^-(x,\phi)} x^-_{jk} = x^+_k. \tag{4}$$

A simple fact is that given $\{x^+_k : k \in A^+(x,\phi)\}$, there exists a continuum of sets $\{x^-_{jk} : (k,j) \in A^+(x,\phi) \times A^-(x,\phi)\}$ satisfying (4) such that agent $j \in A^-(x,\phi)$ has $x_j + x^-_j$, where

$$x^-_j = \textstyle\sum_{k\in A^+(x,\phi)} x^-_{jk} \text{ and } p(\phi)(x_j + x^-_j) = p(\phi)\phi_j, \tag{5}$$

following the transfer.[4] Picking any $\{x^-_j : A^-(x,\phi)\}$, we construct a candidate sequence of allocations from $\{z(\pi; x, \phi) : 0 \le \pi \le 1\}$ (by picking a finite number of π), where

$$z_i(\pi; x, \phi) = \begin{cases} x_i & \text{if } i \notin A^+(x,\phi) \cup A^-(x,\phi) \\ x_i - \pi x^+_i & \text{if } i \in A^+(x,\phi) \\ x_i + \pi x^-_i & \text{if } i \in A^-(x,\phi) \end{cases}. \tag{6}$$

This is a candidate sequence because $u_i(\phi_i) \ge u_i(z_i(\pi; \phi, x))$ all $i \in A$ and $0 \le \pi \le 1$ and the inequality is weak only if $i \in A^-(x,\phi)$ and $\pi = 1$.[5] Because there is a continuum of sets $\{x^-_j : j \in A^-\}$ satisfying (4) and (5) and any linear combination of two such sets satisfies (4) and (5), strict concavity of u_j implies that there must exist

[4] Given $\{x^+_k\}$, a set of transfers $\{x^-_{jk}\}$ satisfying (4) represents an allocation of the bundle of goods $\sum_{k\in A^+} x^+_k$ among agents in $A^-(x,\phi)$. Because $\sum_{k\in A^+} p(x_k - x^+_k) = \sum_{k\in A^+} p\phi_k$ and $\sum_{i\in A} px = \sum_{i\in A} p\phi$, any allocation of $\sum_{k\in A^+} x^+_k$ has $\sum_{j\in A^-} p(x_j + x^-_j) = \sum_{j\in A^-} p\phi_j$ and, hence, satisfies (5).

[5] Let $z_i = z_i(\pi; \phi, x)$. By construction, $z_i = x_i$ and $u_i(\phi_i) > u_i(z_i)$ for $i \notin A^+(x,\phi) \cup A^-(x,\phi)$. By Assumption 1, $u_k(\phi_k) > u_k(z_k)$ for $k \in A^+(x,\phi)$. Because $(\phi, p(\phi))$ is a WE, (5) implies $u_j(\phi_j) \ge u_j(z_j)$ for $j \in A^-(x,\phi)$ and strict if $\pi < 1$.

a set $\{x_j^-\}$ such that $u_j(\phi_j) > u_j(z_j(1; x, \phi))$ all $j \in A^-(x, \phi)$. For our application, it is convenient to have $x \in \mathbb{R}^{I \times L}_{++}$. Although this need not be the case when $x = \omega$, it can obviously be the case when $x \in X(\omega, p(\phi))$ is sufficiently close to ω. Using Lemma 6, we have the following.

Lemma 9 *Let ω and ϕ satisfy Condition 1. There exists $x(\omega) \in X(\omega, p(\phi)) \cap \mathbb{R}^{I \times L}_{++}$ such that $x(\omega)$ can be reached from ω by two rounds of p-trading with $p = p(\phi)$, $u_i(\phi_i) > u(x_i(\omega))$ all i, and $u_i(\phi_i) > u_i(z_i(\pi; x(\omega), \phi))$ all (i, π).*

Thus the sequence of allocations leading ω to $X(\phi, p(\phi))$ consists of $x(\omega)$ in Lemma 9 and some members from $\{z(\pi; x(\omega), \phi) : 0 \leq \pi \leq 1\}$, including $z(1; x(\omega), \phi)$. What may prevent agents from deviating? The movement from ω to $x(\omega)$ is by two rounds of p-trading. The movement from $x = z(0; x(\omega), \phi)$ to $z(1; x(\omega), \phi)$ is around members of $\{z(\pi; x(\omega), \phi) : 0 \leq \pi \leq 1\}$ with restricted trading volumes in each round of trading—one cannot gain much from a deviation if he cannot buy or sell much.[6]

Lemma 10 *Let ω and ϕ satisfy Condition 1. Let $z(\pi) = z(.; x(\omega), \phi)$. There exists a sequence of profiles of trading actions $\{\gamma(r)\}_{r=1}^R$ and allocations $\{y(r)\}_{r=1}^R$ such that $y(1) = z(0)$, $\gamma(r) \in \Gamma(y(r))$ for $1 \leq r \leq R$, $y(r+1) = \lambda(y(r), \gamma(r))$ for $1 < r \leq R-1$, and $z(1) = \lambda(y(R), \gamma(R))$. Moreover, if $y(r)$ is the allocation before trading and all agents other than i take trading actions given by $\gamma(r)$, then the allocation y' after trading satisfies $u_i(\phi_i) > u_i(y_i')$.*

Proof. Let $x = x(\omega)$, $A^+ = A^+(x, \phi)$, and $A^- = A^-(x, \phi)$. We first show that starting from $z(\pi)$ for any $\pi \in [0, 1)$, agents can move to $z(\pi + \delta)$ for any $\delta \in (0, 1 - \pi)$ by finite rounds of trade. To this end, fix $(k, j) \in A^+ \times A^-$, $\epsilon_j \in (0, 0.5x_{j1})$, $m \geq 2$, and $\iota_k \in (0, x_{km})$. We describe trading actions to accomplish the transfer of δx_{jk}^- by 3 consecutive rounds of trade so that it takes $3\#A^+ \times A^-$ rounds to move from $z(\pi)$ to $z(\pi + \delta)$.

Round 1. Agent j spends $\delta\epsilon_j$ units of cash to buy $\delta\iota_k$ units of good m from agent k. Formally, $\gamma_{km} = (\delta\iota_k, 0, p_m)$, $\gamma_{jm} = (0, \delta\epsilon_j, p_m)$, and γ_{im} is inactive for $i \neq k, j$, where $p_m = \epsilon_j/\iota_k$; γ_{il} is inactive for $l \neq m$, all i.

[6] Significance of $u_j(\phi_j) > u_j(z_j)$, where $z = z(1; x, \phi)$, can be seen by letting j be the first agent in $A^-(x, \phi)$ to reach z_j. If $u_j(\phi_j) = u_j(z_j)$, then after j reaches z_j, he may obtain some endowment better than ϕ_j in the remaining trading for the wealth transfer regardless of how the trading volumes are restricted.

Round 2. Agent j spends $\delta\epsilon_j$ units of cash to buy the bundle $(\delta x^-_{kj2}, ..., \delta x^-_{kjL})$ from agent k. Formally, for each $l \geq 2$, $\gamma_{kl} = (\delta x^-_{kjl}, 0, p_l)$, $\gamma_{jl} = (0, p_l \delta x^-_{kjl}, p_l)$, and γ_{il} is inactive for $i \neq k, j$, where $\sum_{l\geq 2} p_l x^-_{kjl} = \delta\epsilon_j$ and $p_l > 0$ all l.

Round 3. Agent k spends $2\delta\epsilon_j + x^-_{kj1}$ units of cash to buy $\delta\iota_k$ units of good m from agent j. Formally, $\gamma_{km} = (0, 2\delta\epsilon_j + x^-_{kj1}, p_m)$, $\gamma_{jm} = (\delta\iota_k, 0, p_m)$, and γ_{im} is inactive for $i \neq k, j$, where $p_m = (2\epsilon_j + x^-_{kj1}/\delta)/\iota_k$; γ_{il} is inactive for $l \neq m$, all i.

Next, we show that there exists $\bar{\delta} \in (0, 1)$ independent of π such that for any $\delta \leq \bar{\delta}$ and any i, if y is the allocation before a round of trading during the movement from $z(\pi)$ to $z(\pi + \delta)$ and if all agents other than i take trading actions described above, then the allocation y' after trading in that round satisfies $u_i(\phi_i) > u_i(y'_i)$. For this, note $v_i(\bar{\delta}_{kj}) = \max u_i(z_i(\pi') + \delta x^-_{kj} + \delta(2\epsilon_j + \iota_k, ..., 2\epsilon_j + \iota_k))$ subject to $\pi' \in [0, 1]$ and $0 \leq \delta \leq \bar{\delta}_{kj}$ is well defined for any $\bar{\delta}_{kj} \geq 0$. Because $u_i(z_i(\pi')) < u_i(\phi_i)$ all $\pi' \in [0, 1]$, the theorem of maximum implies that there exists $\bar{\delta}_{ikj}$ such that $v_i(\bar{\delta}_{kj}) < u_i(\phi_i)$. When y is the allocation before trading in any of the above three rounds, y'_i is bounded above by $z_i(\pi) + \delta x^-_{kj} + \delta(2\epsilon_j + \iota_k, ..., 2\epsilon_j + \iota_k)$. So we can set $\bar{\delta} = \min\{\bar{\delta}_{ikj} : (k, j) \in A^+ \times A^-, i \in A\}$.

Finally, let $\delta \leq \bar{\delta}$ and $1/\delta$ be an integer. Then agents can move from $z(0)$ to $z(1)$ by $R = (3/\delta)\#A^+ \times A^-$ rounds of trading. ■

4.4 Completion of proof

There are two cases in Theorem 2: (i) Condition 1 is satisfied when $(\omega, \phi) = (\omega^0, \phi^0)$ and (ii) $\phi^0 = c(\omega^0)$. The equilibrium for case (i) can be thought as a loop. For this loop, the input of each step is some (ω, ϕ) satisfying Condition 1; the input of the initial step is $(\omega, \phi) = (\omega^0, \phi^0)$. In the current step, agents move from ω to ϕ by two substeps: from ω to some $z \in X(\phi, p(\phi))$ at substep 1 and from z to ϕ at substep 2. The substep-1 movement follows the path given by Lemmas 9 and 10; the substep-2 movement is by p-trading with $p = p(\phi)$. If no agent deviates in the two-substep movement of the current step, then the loop stops. If agent i deviates and results in an allocation ω', then by our design of the movement, ω'_i must be strictly dominated by ϕ_i; now we let the output of the current step and the input for the next step as (ω', ϕ'), where ϕ' is z'' in Lemma 8 with $(z', c(z)) = (\omega', \phi)$. For case (ii), we can simply let the above loop skip substep 1 in the initial step and identify z with ω at substep 2 in the initial step. Therefore, the proof for one case implies the proof for another case. Now we provide a formal and complete description of the equilibrium for case (i); the result for case (ii) follows as a corollary.

Proposition 1 *If Condition 1 is satisfied when* $(\omega, \phi) = (\omega^0, \phi^0)$*, then* ϕ^0 *is the outcome of the game specified by some equilibrium* f*.*

Proof. To simplify exposition, we present the proof here as if there were no test-for-ending rounds and the game were over right reaching an efficient allocation; see the appendix for the treatment that includes test-for-ending rounds.

To describe f, we first encode histories into states. The set of all states is denoted by Θ. We classify states of Θ into different subsets, each of which is indexed by a pair of (ω, ϕ) satisfying Condition 1 and denoted by $\kappa(\omega, \phi)$. The subset $\kappa(\omega, \phi)$ itself consists of three subsets, $\kappa(0, \omega, \phi)$, $\kappa(1, \omega, \phi)$, and $\kappa(2, \omega, \phi)$. Specifically, $\kappa(0, \omega, \phi) = \{\omega, x'(\omega)\}$, where $x'(\omega)$ is an allocation which is close to ω, can be reached from ω by one round of p-trading, and can reach $x(\omega)$ in Lemma 10 by another round of p-trading with $p = p(\phi)$; $\kappa(1, \omega, \phi)$ is the set of allocations $\{y(r)\}_{r=1}^{R}$ defined by Lemma 10 for given (ω, ϕ); and $\kappa(2, \omega, \phi) = X(\phi, p(\phi)) \setminus \{\phi\}$. Notice that we represent a state of $\kappa(n, \omega, \phi)$ for $n \in \{0, 1, 2\}$ by an allocation; thus the state $\theta \in \Theta$ that corresponds to an allocation x of $\kappa(n, \omega, \phi)$ may be equivalently expressed as (x, n, ω, ϕ). Below we denote the allocation associated to a state $\theta \in \Theta$ by θ and when we identify an allocation with a state, we indicate to which subset of Θ the allocation belongs.

Next we introduce two mappings, α and η. The mapping $\alpha = (\alpha_1, ..., \alpha_I)$ assigns a trading action $\alpha_i(\theta) \in \Gamma_i(\theta_i)$ for agent i at the current trading round when the current state is θ. The mapping η assigns to (θ, γ) a state $\eta(\theta, \gamma) \in \Theta$ as the next state given that γ is the profile of actions taken by agents at the current state θ.

(i) $\theta \in \kappa(0, \omega, \phi)$. When $\theta = \omega$, $\alpha(\theta)$ is the p-trading profile by which agents can reach $x'(\omega)$ from ω; $\eta(\theta, \alpha(\theta))$ is $x'(\omega)$ of $\kappa(0, \omega, \phi)$. When $\theta = x'(\omega)$, $\alpha(\theta)$ is the p-trading profile by which agents can reach $x(\omega)$ from $x'(\omega)$; $\eta(\theta, \alpha(\theta))$ is $y(1)$ of $\kappa(1, \omega, \phi)$.

(ii) $\theta \in \kappa(1, \omega, \phi)$. Given $\theta = y(r)$, $\alpha(\theta)$ is $\gamma(r)$ defined by Lemma 10 for (ω, ϕ); $\eta(\theta, \alpha(\theta))$ is $y(r+1)$ of $\kappa(1, \omega, \phi)$ if $r < R$ and is $y(r+1)$ of $\kappa(2, \omega, \phi)$ if $r = R$.

(iii) $\theta \in \kappa(2, \omega, \phi)$. Now $\alpha(\theta)$ is the p-trading profile $a(\theta)$ given by Definition 2 with $p = p(\phi)$; $\eta(\theta, \alpha(\theta))$ is $\lambda(\theta, \alpha(\theta))$ of $\kappa(2, \omega, \phi)$.

For $\eta(\theta, \gamma)$ with $\gamma \neq \alpha(\theta)$, in case $\gamma_i \neq \alpha_i(\theta)$ for only one i, if $\theta \in \kappa(2, \omega, \phi)$ and $\lambda(\theta, \gamma) \in X(\phi, p(\phi))$, then $\eta(\theta, \gamma)$ is $\lambda(\theta, \gamma)$ of $\kappa(2, \omega, \phi)$; otherwise, $\eta(\theta, \gamma)$ is $\lambda(\theta, \gamma)$

of $\kappa(0, \omega', \phi')$, where ω' is $\lambda(\theta, \gamma)$ and ϕ' is z'' of Lemma 8 with $(z', c(z)) = (\omega', \phi)$. Other cases of $\eta(\theta, \gamma)$ are dealt with in the appendix.

The set Θ and mappings α and η completely describe the candidate equilibrium f. Let the initial state be ω^0 of $\kappa(0, \phi^0, \omega^0)$. If all agents follow actions specified by α and the transition of states specified by η, then the outcome of the game is ϕ^0. Now fix i and we show that starting from any state $\theta \in \kappa(\omega, \phi)$, i cannot benefit from a unilateral deviation. Note that if i does not deviate, he consumes ϕ_i. Also, note that there are three sorts of deviations: (a) Deviating from f_i by a finite number of times and the state following the last deviation is $\theta' \notin \kappa(\omega, \phi)$;[7] (b) Deviating from f_i by a finite number of times and the state following the last deviation is $\theta' \in \kappa(\omega, \phi)$; (c) Deviating from f_i by an infinite number of times. Taking a sort-c deviation, i consumes 0. Taking a sort-b deviation, i consumes ϕ_i. Taking a sort-a deviation, i consumes some ϕ_i' such that $u_i(\phi_i) > u_i(\phi_i')$. This completes the proof. ■

Corollary 1 *If $\phi^0 = c(\omega^0)$, then ϕ^0 is the outcome of the game specified by some equilibrium f.*

5 The concluding remarks

We have shown that costless retrading drives out inefficiency and sustains a large set of efficient allocations in a dynamic version of the SS market game when individuals have market power. The large set of efficient allocations contain WAs for the initial allocation. On-path trading to reach a WA does not rely on non-anonymous information and in some special case anonymous information is sufficient for off-path plays; on-path trading to reach a non-WA relies on non-anonymous information.

Our results apply to any finite population size. But may the population size be a factor? Consider the N-replica of the economy. Suppose people stick to p-trading in each round of trade, where p is a WP for the initial allocation. When N increases, the benefit from any possible advantageous redistribution decreases, and the cost for other agents to ignoring non-anonymous information decreases, too. In fact, given ϵ there exists N_ϵ such that if $N > N_\epsilon$ then the sticking to p-trading is an ϵ equilibrium.

[7] We do not apply the one-step deviation principle here because there is no discounting.

Appendix

Proof of Lemma 1

Fix ω and we proceed by three steps. First, we show that given γ and a market price $p_l > 0$, the set of excess demand for each good is well defined. Next, we show that with the rationing scheme, there is a well-defined market-clearing price $p_l(\gamma)$ for each good. Finally, we show that given γ and $p_l(\gamma)$, the trade for each agent in trading post l is defined and so is $\lambda(\omega, \gamma)$.

Step 1. In the present-round trading post l, the market demand (buying) curve and the supply (offering) curve are

$$Q_l^0(p_l) = [\sum_{i \in A_{l++}} b_{il}/p_l, \sum_{i \in A_{l+}} b_{il}/p_l] \text{ and } Q_l^1(p_l) = [\sum_{i \in A_{l--}} s_{il}, \sum_{i \in A_{l-}} s_{il}], \tag{7}$$

respectively, where $A_{l++} = \{i : p_{il} > p_l\}$, $A_{l+} = \{i : p_{il} \geq p_l\}$, $A_{l--} = \{i : p_{il} < p_l\}$, and $A_{l-} = \{i : p_{il} \leq p_l\}$. So the set of excess demand is

$$D_l(p_l) = \left\{s | \exists s^0 \in Q_l^0(p_l),\ s^1 \in Q_l^1(p_l),\ s = s^0 - s^1\right\}. \tag{8}$$

Step 2. Let

$$p_l(\gamma) = \begin{cases} 0.5(p_l^1 - p_l^0) & p_l^0 < p_l^1 \\ 0 & p_l^0 = p_l^1 = 0 \\ \{p_l : 0 \in D_l(p_l)\}, & \text{otherwise} \end{cases},$$

where

$$p_l^0 = \begin{cases} \max\limits_{b_{il} > 0} p_{il} & \text{if some } b_{il} > 0 \\ 0 & \text{otherwise} \end{cases}, \quad p_l^1 = \begin{cases} \min\limits_{s_{il} > 0} p_{il} & \text{if some } s_{il} > 0 \\ \infty & \text{otherwise} \end{cases}.$$

(Note that, when $p_l^0 < p_l^1$, for example, when sellers of good l offer nothing and buyers want to spend some numeraire on good l, $p_l(\gamma) = 0.5(p_l^1 - p_l^0)$ is one of the prices that lead to no trade in the market for good l.) To complete step 2, we claim that given $\gamma \in \Gamma$, if $p_l^0 \geq p_l^1$ and $p_l^1 > 0$, then there exists a unique $p_l(\gamma) > 0$ such that $0 \in D_l(p_l(\gamma))$ for each $2 \leq l \leq L$. The claim is verified below.

Step 3. To determine how much good l each agent buys or sells, we need to take care of the situation where there is excess demand or excess supply at $p_l(\omega, \gamma)$. Let

$$D_l^0 = \max\left\{\sup Q_l^0(p_l(\gamma)) - \inf Q_l^1(p_l(\gamma)), 0\right\},$$

$$D_l^1 = \max\left\{\sup Q_l^1(p_l(\gamma)) - \inf Q_l^0(p_l(\gamma)), 0\right\},$$

$$B_l = \sum_{A_l} b_{il},\ S_l = \sum_{A_l} s_{il},\ A_l = \{i : p_{il} = p_l(\gamma)\};$$

notice that D_l^0 and D_l^1 are the residual demand for sellers of good l and residual supply for buyers of good l, respectively, who quote the price equal to $p_l(\gamma)$. Then the amount of good l bought by agent i is

$$\psi_{il}(\gamma) = \begin{cases} b_{il}/p_l(\gamma) & \text{if } p_{il} > p_l(\gamma) > 0 \\ \min\left[b_{il}D_l^1/B_l, b_{il}/p_l(\gamma)\right] & \text{if } p_{il} = p_l(\gamma) > 0 \text{ and } b_{il} > 0 \\ b_{il}S_l/B_l & \text{if } p_{il} = p_l(\gamma) = 0 \text{ and } b_{il} > 0 \\ 0 & \text{otherwise} \end{cases}, \tag{9}$$

and the amount of good l sold by agent i is

$$\varphi_{il}(\gamma) = \begin{cases} s_{il} & \text{if } p_{il} < p_l(\gamma) \\ s_{il} & \text{if } p_{il} = p_l(\gamma) = 0 \text{ and } b_{jl} > 0 \text{ for some } j \\ \min\left[s_{il}D_l^0/S_l,\ s_{il}\right] & \text{if } p_{il} = p_l(\gamma) > 0 \text{ and } s_{il} \neq 0 \\ 0 & \text{otherwise} \end{cases}. \tag{10}$$

Equations (9) and (10) imply that all buyers who quote a price higher than $p_l(\gamma)$ in their strategies spend the amounts of the numeraire they quote to purchase good l. All sellers who quote a price lower than $p_l(\gamma)$ sell the quantities of good l they quote. If there is an excess demand or an excess supply of good l at $p_l(\gamma)$, those buyers and sellers who quote the price $p_l(\gamma)$ are rationed in proportion to their announced amounts of the numeraire to be spent on good l and supplies of good l, respectively. Other buyers and sellers do not trade in the market for good l. It follows that

$$\lambda_{il}(\omega, \gamma) = \omega_{il} + \psi_{il}(\gamma) - \varphi_{il}(\gamma),\ 2 \leq l \leq L, \tag{11}$$

and

$$\lambda_{i1}(\omega,\gamma) = \omega_{i1} + \sum_{2\le l\le L}[\varphi_{il}(\gamma) - \psi_{il}(\gamma)]p_l(\gamma) \tag{12}$$

Now we verify the claim at the end of step 2. As is standard, both $Q_l^0(p_l)$ and $Q_l^1(p_l)$ (see (7)) are upper hemicontinuous and convex valued and so is $D_l(p_l)$ (see (8)). Monotonicity of $Q_l^0(.)$ ($Q_l^1(.)$, resp.) is defined according to comparing two elements of $Q_l^0(p_l)$ and $Q_l^0(p_l')$ ($Q_l^1(p_l)$ and $Q_l^1(p_l')$, resp.) for any p_l and p_l'. By this definition, $Q_l^1(p_l)$ is non-decreasing in p_l, $Q_l^0(p_l)$ is strictly decreasing in p_l, and $D_l(p_l)$ is strictly decreasing in p_l. Let

$$P_l = \{p_l : \exists d_p \in D_l(p_l) \text{ and } d_p \le 0\}.$$

To see P_l is not empty, choose any $p_l > p_l^0$. For any $s^0 \in Q_l^0(p_l)$, $s^0 = 0$; by assumption, $p_l > p_l^0 \ge p_l^1 > 0$, for any $s^1 \in Q_l^1(p_l)$, $s^1 \ge 0$. Hence $p_l \in P_l$. A candidate for $p_l(\gamma)$ is $\varrho \equiv \inf P_l$. Apparently, $\varrho < \bar{\varrho} \equiv p_l^0 + 1 \in P_l$. Let

$$\underline{\varrho} = 0.5\sum_{i\in A_l^0} b_{il} / \sum_{1\le j\le I}\omega_{jl},\ A_l^0 = \{i : p_{il} = p_l^0\}.$$

Note that for any $d_p \in D(\underline{\varrho})$, $d_p > 0$. Hence, $\varrho \ge \underline{\varrho} > 0$.

If $0 \in D_l(\varrho)$, then we are done. If $0 \notin D_l(\varrho)$, we claim that there is $d_1 > 0$ and $d_2 < 0$ such that d_1, $d_2 \in D_l(\varrho)$. Then by the convexity of $D_l(p_l)$, $0 \in D_l(\varrho)$. By definition, $\forall p_l < \varrho$, $0 \notin D_l(p_l)$. Since $\inf D_l(\varrho) \le 0$, by monotonicity, $\forall p_l > \varrho$, and $\forall d \in D_l(p_l)$, $d < \inf D_l(\varrho) \le 0$. This proves the uniqueness of $p_l(\gamma)$.

To see that there is $d_1 > 0$ and $d_2 < 0$ such that d_1, $d_2 \in D_l(\varrho)$ when $0 \notin D_l(\varrho)$, first suppose by contradiction that $\forall\ d \in D_l(\varrho)$, $d < 0$. By definition of ϱ, we can choose p_l that satisfies $\underline{\varrho} \le p_l < \varrho$, there is no $d_p \in D_l(p_l)$ such that $d_p \le 0$. Now consider a sequence $\{p^n\}$ of price such that $\underline{\varrho} \le p^n < \varrho$ and $p^n \to \varrho$ as $n \to \infty$. Then there is $d \in D_l(p^n)$ with $d > 0$. Choose any $d_{p^n} \in D_l(p^n)$ with $d_{p^n} > 0$. Because $D_l(p^n)$ and $\{d_{p^n}\}$ are bounded, there is a subsequence of $\{p^n\}$ whose corresponding subsequence of $\{d_{p^n}\}$ converges. Obviously, this subsequence of $\{d_{p^n}\}$ converges to a nonnegative limit. Then by upper hemicontinuity of $D_l(p_l)$, $d \in D_l(\varrho)$, a contradiction. Next suppose by contradiction that $\forall d \in D_l(\varrho)$, $d > 0$. Consider a sequence $\{p^n\}$ of price such that $\bar{\varrho} \ge p^n > \varrho$ and $p^n \to \varrho$ as $n \to \infty$. Since $D_l(p_l)$ is decreasing, there is $d \in D_l(p^n)$ with $d \le 0$. Take any $d_{p^n} \in D_l(p^n)$ such that $d_{p^n} \le 0$. Because $D_l(p^n)$ and $\{d_{p^n}\}$ are bounded, there is a subsequence of $\{p^n\}$ whose corresponding subsequence of $\{d_{p^n}\}$ converges. Obviously, this subsequence of $\{d_{p^n}\}$ converges to a nonpositive limit. Then by upper hemicontinuity of $D_l(p_l)$, $d \in D_l(\varrho)$, a contradiction.

Completion of proof of Lemma 4

As indicated in the main text, Lemma 4 is valid for any W when $I = 2$. So here we consider $I \geq 3$. Differentiability of the optimal demand function $q_{jl}(\vartheta_i, \vartheta_i x_j)$ is standard. After linear transformation, the Jacobian of Φ at any (x, ϑ, w) can be written as

$$\begin{bmatrix}
\begin{bmatrix}
\bar{\mathbf{0}} & \vdots & \frac{\partial \bar{q}_2^{(1)}}{\partial x_2} & \cdots & \frac{\partial \bar{q}_I^{(1)}}{\partial x_I} \\
\cdots & \ddots & \cdots & \cdots & \cdots \\
\frac{\partial \bar{q}_2^{(2)}}{\partial x_2} & \vdots & \bar{\mathbf{0}} & \cdots & \frac{\partial \bar{q}_I^{(2)}}{\partial x_I} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial \bar{q}_1^{(I)}}{\partial x_1} & \vdots & \frac{\partial \bar{q}_2^{(I)}}{\partial x_2} & \cdots & \bar{\mathbf{0}}
\end{bmatrix}
& \vdots &
\begin{bmatrix}
\frac{\partial q_j^{(1)}}{\partial \vartheta_1} & \vdots & \mathbf{0}' & \cdots & \mathbf{0}' \\
\cdots & \ddots & \cdots & \cdots & \cdots \\
\mathbf{0}' & \vdots & \frac{\partial q_j^{(2)}}{\partial \vartheta_2} & \cdots & \mathbf{0}' \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0}' & \vdots & \mathbf{0}' & \cdots & \frac{\partial q_j^{(I)}}{\partial \vartheta_I}
\end{bmatrix}
& \vdots &
\begin{bmatrix}
\bar{\mathbf{0}} \\ \cdots \\ \bar{\mathbf{0}} \\ \cdots \\ \vdots \\ \bar{\mathbf{0}}
\end{bmatrix} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\begin{bmatrix}
\frac{\partial \vartheta_1}{\partial x_1} & \vdots & \mathbf{0} & \cdots & \mathbf{0} \\
\cdots & \ddots & \cdots & \cdots & \cdots \\
\mathbf{0} & \vdots & \frac{\partial \vartheta_2}{\partial x_2} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \vdots & \mathbf{0} & \cdots & \frac{\partial \vartheta_I}{\partial x_I}
\end{bmatrix}
& \vdots &
\begin{bmatrix}
\hat{\mathbf{I}} & \vdots & \mathbf{0} & \vdots & \cdots & \mathbf{0} \\
\cdots & \ddots & \cdots & \vdots & \cdots & \cdots \\
\mathbf{0} & \vdots & \hat{\mathbf{I}} & \vdots & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \vdots & \mathbf{0} & \vdots & \cdots & \hat{\mathbf{I}}
\end{bmatrix}
& \vdots &
\begin{bmatrix}
\mathbf{0} \\ \cdots \\ \mathbf{0} \\ \cdots \\ \vdots \\ \mathbf{0}
\end{bmatrix} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\begin{bmatrix} \mathbf{0} & \vdots & \mathbf{0} & \vdots & \cdots & \mathbf{0} \end{bmatrix}
& \vdots &
\begin{bmatrix} \mathbf{0}' & \vdots & \mathbf{0}' & \vdots & \cdots & \mathbf{0}' \end{bmatrix}
& \vdots & \mathbf{I}
\end{bmatrix}. \tag{13}$$

In (13), $\mathbf{I}$ is the $L \times L$ identify matrix, $\hat{\mathbf{I}}$ is the $(L-1) \times (L-1)$ identify matrix, $\mathbf{0}$ is the $(L-1) \times L$ zero matrix, $\mathbf{0}'$ is the $L \times (L-1)$ zero matrix, and $\bar{\mathbf{0}}$ the $L \times L$ zero matrix. Moreover,

$$\frac{\partial \bar{q}_j^{(i)}}{\partial x_j} = \begin{bmatrix}
q_{j1\kappa}^{(i)} - 1 & \cdots & \vartheta_{iL} q_{jL\kappa}^{(i)} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
q_{jL\kappa}^{(i)} & \cdots & \vartheta_{iL} q_{jL\kappa}^{(i)} - 1
\end{bmatrix}, \frac{\partial \vartheta_i}{\partial x_i} = \begin{bmatrix}
\frac{\partial U_{i2}(x_i)}{\partial x_{i1}} & \cdots & \frac{\partial U_{i2}(x_i)}{\partial x_{iL}} \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
\frac{\partial U_{iL}(x_i)}{\partial x_{i1}} & \cdots & \frac{\partial U_{iL}(x_i)}{\partial x_{iL}}
\end{bmatrix},$$

and

$$\frac{\partial q_j^{(i)}}{\partial \vartheta_i} = \begin{bmatrix} \sum\limits_{j\neq i} \frac{\partial q_{j1}(\vartheta_i, x_j\vartheta_i)}{\partial \vartheta_{i2}} & \cdots & \sum\limits_{j\neq i} \frac{\partial q_{j1}(\vartheta_i, x_j\vartheta_i)}{\partial \vartheta_{iL}} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \sum\limits_{j\neq i} \frac{\partial q_{jL}(\vartheta_i, x_j\vartheta_i)}{\partial \vartheta_{iI}} & \cdots & \sum\limits_{j\neq i} \frac{\partial q_{jL}(\vartheta_i, x_j\vartheta_i)}{\partial \vartheta_{iL}} \end{bmatrix},$$

where $q_{jl\kappa}^{(i)}$ is the partial derivative of the optimal demand function $q_{jl}(\vartheta_i, \vartheta_i x_j)$ with respect to wealth $x_j\vartheta_i$. We have

$$v_j^{(i)\prime}(\kappa) = u_{jl}(q_j^{(i)})q_{jl\kappa}^{(i)}, \tag{14}$$

where $v_j^{(i)}(\kappa)$ is the maximal utility of agent j under the price ϑ_i when his wealth is κ.

The matrix in (13) is invertible at $(x, \vartheta, w) \in F$ if for every $1 \leq j \leq I$, its columns $(j-1)L+1$ to jL are linear independent. Consider $j = 1$ and it suffices to show that

$$\begin{bmatrix} \frac{\partial \bar{q}_1^{(2)}}{\partial x_1} \\ \vdots \\ \frac{\partial \bar{q}_1^{(I)}}{\partial x_1} \end{bmatrix}$$

has the full rank L. (As is well known, the rank of $\frac{\partial \bar{q}_2^{(i)}}{\partial x_2}$ is $L-1$ for each $i \geq 2$.) Because x is inefficient, $(x, \vartheta, w) \in F$ requires that the set $\{\vartheta_i : 1 \leq i \leq I\}$ contain at least 3 distinct values. Without loss of generality, suppose $\vartheta_2 \neq \vartheta_3$. Suppose by contradiction that the rank of $\begin{bmatrix} \frac{\partial \bar{q}_1^{(2)}}{\partial x_1} \\ \frac{\partial \bar{q}_1^{(3)}}{\partial x_1} \end{bmatrix}$ at $L-1$. Then it is necessary to have

$$q_{1l\kappa}^{(2)} = q_{1l\kappa}^{(3)} \sum_{m\geq 1} \vartheta_{3m} q_{1m\kappa}^{(2)} \tag{15}$$

all l. By (14), (15) implies $u_{1l}(q_1^{(3)}) = C \cdot u_{1l}(q_1^{(2)})$ all l for some constant C so that

$$\vartheta_{2l} = U_{1l}(q_1^{(2)}) = \frac{u_{1l}(q_1^{(2)})}{u_{11}(q_1^{(2)})} = \frac{u_{1l}(q_1^{(3)})}{u_{11}(q_1^{(3)})} = U_{1l}(q_1^{(3)}) = \vartheta_{3l}$$

all l, which is a contradiction. This completes the proof.

Proof of Lemma 8

Let $x' = \arg\max_{x\in\Omega} \sum_{j\neq i} u_j(x_j)$ s.t. $u_j(x_j) \geq u_j(z'_j)$ for $j \neq i$ and $u_i(x'_i) = u_i(z'_i)$. By definition, x' is efficient so $u_k(x'_k) > u_k(z'_k)$ for some $k \neq i$. By slightly reducing the endowment of agent k from x'_k and redistributing among all other agents, we obtain an allocation x'' with $u_j(x''_j) > u_j(z'_j)$ all $j \neq i$ and $u_i(c_i(z)) > u(x''_i) > u_i(z'_i)$. Now let $z'' = \arg\max_{x\in\Omega} \sum_{j\neq i} u_j(x_j)$ s.t. $u_j(x_j) \geq u_j(x''_j)$ for $j \neq i$ and $u_i(x_i) = u_i(x''_i)$.

Completion of proof of Proposition 1

To accommodate test-for-ending rounds, we add to the set Θ an additional subset whose states correspond to histories that lead to test-for-ending round. The subset, denoted ι, consists of all members of Ω. Moreover, we add to the subset $\kappa(2, \omega, \phi)$ for each indexing pair (ω, ϕ) an additional member, namely, ϕ. When ϕ is in $\kappa(2, \omega, \phi)$, $\alpha_i(\phi)$ is a trading action with all inactive orders; $\eta(\phi, \alpha(\phi))$ is ϕ of ι. For $\eta(\phi, \gamma)$ with $\gamma \neq \alpha(\phi)$, in case $\gamma_i \neq \alpha_i(\theta)$ for only one i, if $\lambda(\phi, \gamma) \in X(\phi, p(\phi))$, then $\eta(\phi, \gamma)$ is $\lambda(\phi, \gamma)$ of $\kappa(2, \omega, \phi)$; otherwise, $\eta(\phi, \gamma)$ is $\lambda(\phi, \gamma)$ of $\kappa(0, \omega', \phi')$, where ω' is $\lambda(\phi, \gamma)$ and ϕ' is z'' in Lemma 8 with $(z', c(z)) = (\omega', \phi)$. Other cases of $\eta(\phi, \gamma)$ are dealt with below.

To describe how the mappings α and η act on states of ι, we need some additional notation. Let $\zeta_{1i} = \{(s_{il}, b_{il}, p_{il}) : l \in K_i\}$ and $\zeta_{2i} = \{(s_{il}, b_{il}, p_{il}) : l \notin K_i\}$ denote a set of active orders and a set of orders, respectively, submitted by agent i at stage 1 and at stage 2 of a test-for-ending such that $\zeta_i = (\zeta_{1i}, \zeta_{2i})$ is a trading action of $\Gamma_i(x_i)$ when x_i is the pre-trading endowment of i. For $\theta \in \iota$, α_i assigns to i some ζ_{1i}, denoted $\zeta_{1i}(\theta)$, and some ζ_{2i} conditional on $\zeta_1 = (\zeta_{11}, ..., \zeta_{1I})$, denoted $\zeta_{2i}(\zeta_1, \theta)$, such that $(\zeta_{1i}(\theta), \zeta_{2i}(\zeta_1, \theta))$ is a trading action of $\Gamma_i(\theta_i)$. And η assigns to $(\theta, \zeta_1, \zeta_2)$ a state $\eta(\theta, \zeta_1, \zeta_2) \in \Theta$ as the next state given that agents submit ζ_1 and ζ_2 at stage 1 and stage 2, respectively, at the current state θ.

Given ζ_1 and the allocation θ, let $G(\zeta_1, \theta)$ denote the static game in which agent i chooses to submit (s_{il}, b_{il}, p_{il}) for l not in the set K_i associated to ζ_{1i} such that ζ_i constituted by the given ζ_{1i} and the chose ζ_{2i} is a trading action of $\Gamma_i(\theta_i)$. When the choices of all agents give rise to a trading profile ζ of $\Gamma(\theta)$, $u_i(\lambda_i(\theta, \zeta))$ is the payoff of agent i in $G(\zeta_1, \theta)$. Pick a Nash equilibrium of $G(\zeta_1, \theta)$ and denote by $\chi_i(\zeta_1, \theta)$ the orders submitted by agent i in that equilibrium.[8]

[8]Existence of a Nash equilibrium can be obtained by adapting the argument of Simon [21]. For

Now we are ready to give details of $\zeta_{1i}(\theta)$, $\zeta_{2i}(\zeta_1,\theta)$, $\eta(\theta,\zeta_1,\zeta_2)$ for $\theta \in \iota$. If θ is inefficient then $\zeta_{i1}(\theta)$ consists of orders that sell some good l whenever $\theta_{il} > 0$ and buy some good l whenever $\theta_{il} = 0$ and $\theta_{i1} > 0$ under the reservation price $p_l(\theta)$ for each $l \geq 2$; if θ is efficient then $\zeta_{i1}(\theta)$ is empty (K_i is empty); and $\zeta_{i2}(\zeta_1,\theta)$ is always $\chi_i(\zeta_1,\theta)$. If $\lambda(\theta,\zeta)$ is efficient but ζ has at least one active order or if $\lambda(\theta,\zeta)$ is inefficient and $\lambda(\theta,\zeta) \in X(\theta,p(\theta))$, then $\eta(\theta,\zeta_1,\zeta_2)$ is $\lambda(\theta,\zeta)$ of $\kappa(2,\omega,\phi)$, where $\phi = c(\theta)$ (ω can be arbitrary as long as Condition 1 is satisfied). If $\lambda(\theta,\zeta)$ is inefficient, $\lambda(\theta,\zeta) \notin X(\theta,p(\theta))$, and either $\zeta_{1i} \neq \zeta_{1i}(\theta)$ for only one i and $\zeta_2 = \zeta_2(\zeta_1,\theta)$ or $\zeta_{2i} \neq \zeta_{2i}(\zeta_1,\theta)$ for only one i, then $\eta(\theta,\zeta_1,\zeta_2)$ is $\lambda(\theta,\gamma)$ of $\kappa(0,\omega,\phi)$, where ω is $\lambda(\theta,\gamma)$ and ϕ is z'' in Lemma 8 with $(z',z) = (\omega,\theta)$. Other cases of $\eta(\theta,\zeta_1,\zeta_2)$ are dealt with below.

Next we show that starting from any state θ of ι, i cannot benefit from a unilateral deviation. When θ is inefficient, because each agent $j \neq i$ submits an active order at each trading post at stage 1 provided that j has the resource to do so, the two-stage submission does not affect Lemma 7. So if i does not deviate, then he consumes $c_i(\theta)$; otherwise, his consumption cannot exceed $c_i(\theta)$. When θ is efficient, if i does not deviate, then he consumes θ_i; otherwise, his consumption cannot exceed θ_i.

Finally, we provide details of the mapping η for the cases that are dealt with above. First consider θ of $\kappa(n,\omega,\phi)$ and $\gamma_i \neq \alpha_i(\theta)$ for more than one i. If γ is not a profile with all inactive orders, $\eta(\theta,\gamma)$ is $\lambda(\theta,\gamma)$ of $\kappa(2,\omega',\phi')$, where $\phi' = c(\lambda(\theta,\gamma))$ (ω' can be arbitrary as long as Condition 1 is satisfied); otherwise, $\eta(\theta,\gamma)$ is $\lambda(\theta,\gamma)$ of ι. Next consider θ of ι, $\lambda(\theta,\zeta)$ is inefficient, $\lambda(\theta,\zeta) \notin X(\theta,p(\theta))$, either $\zeta_{1i} \neq \zeta_{1i}(\theta)$ for more than one i and $\zeta_2 = \zeta_2(\zeta_1,\theta)$ or $\zeta_{2i} \neq \zeta_{2i}(\zeta_1,\theta)$ for more than one i, and ζ is not a profile with all inactive orders. Now $\eta(\theta,\zeta_1,\zeta_2)$ is $\lambda(\theta,\gamma)$ of $\kappa(0,\omega,\phi)$, where ω is $\lambda(\theta,\gamma)$ and ϕ is z'' in Lemma 8 with $(z',z) = (\omega,\theta)$.

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agent i, there is a constrained demand problem for the types of goods not in K_i analogous to the constrained demand problem in Simon [21]. A constrained Walrasian price is a price vector under which the net demand for each good $l \geq 2$ deriving from the solutions of the stage-2 constrained problems and some selection of the net demand correspondence implied by ζ_1 (see (8)) is equal to zero. Existence of the constrained Walrasian price uses the fact that all stage-1 net demand correspondences are upper hemicontinuous and convex-valued. Then one can construct each agent's strategy in $G(\zeta_1,\theta)$ based on the constrained Walrasian equilibrium.

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