On Nonneutrality of the Exchange-Rate Regime*

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Abstract

The presumption that flexible internal prices render the exchange-rate regime neutral in the presence of the aggregate shocks overlooks the channel that the current nominal quantity of a currency may affect current output through its (expected and real) future value. Using a two-country variant of the Lagos-Wright model, we demonstrate that this channel undermines the neutrality presumption. We further explore which regime better arranges the future values of currencies to sustain efficient output in setups where efficiency requires policy intervention (the Friedman rule) but policy running is endogenously costly and where efficiency may be attained without policy intervention.

Key words: Flexible prices; exchange-rate regimes; unified currency; optimal currency areas; neutrality

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1 Introduction

There is a presumption that fixed and flexible (floating) exchange-rate regimes are equivalent if prices are flexible:

> If internal prices were as flexible as exchange rates, it would make little economic difference whether adjustments were brought about by changes in exchange rates or equivalent changes in internal prices (Friedman [8, page 165]).)

This view, labeled exchange-rate-regime neutrality or simply neutrality, is the foundation of the literature on optimum currency areas (OCA) and most of the subsequent literature that compares a unified currency (the Euro) to a system of country-specific currencies under flexible exchange rates (see, e.g., Mundell [28] and Krugman [23]). A formal statement of neutrality is in Lucas [25]. For a rationale of neutrality, think of the US and EU, which are subject to some aggregate shocks shifting the demands for the dollar and euro. Suppose today the shock causes a stronger-than-average demand for the dollar. When the exchange rate is flexible, the dollar appreciates above its average while the quantity of each currency is kept at a fixed average. When the exchange rate is fixed, the quantity of dollars rises above its average. Although the fixed exchange rate generates more dollars and less euros than the flexible exchange rate, each economy ends up with the same real balances under the two regimes thanks to flexible internal prices. So the exchange-rate regime is neutral.

The rationale, however, overlooks a potential channel for the quantity of each currency to affect output. How much one dollar or euro can incentivize people who supply labor elastically to work *now* ought to depend on its (expected and real) *future* value. For neutrality to hold in future, this future value ought to be the same under both regimes. But, then, neutrality falls apart——more dollars generated by the fixed exchange rate ought to induce higher *current* labor input and more output in the US after being translated into higher nominal labor earnings. This channel apparently does not present in an exchange-rate model with no production (e.g., the cash-in-advance (CIA) model of Lucas [25] and the money-in-the-utility-function (MUF) model of Obstfeld and Rogoff [33, section 8.7]) or with no currencies (e.g., Gabaix and Maggiori [9] and Itskhoki and Muzhin [15]). The channel ought to function in a model with production and currencies; a simple test is to place the fixed exchange rate in a suitable version of an existing model (e.g., Chari et al. [4] and Cooper and Kempef [6]).

To facilitate deduction of equilibrium properties and to allow the option of endogenizing imperfect substitution of currencies, we study a two-country variant of the Lagos-Wright (LW) [24] model similar to the one in Gomis-Porqueras et al. [10] (who focus on numerical analysis of flexible exchange rates). In our model, each country produces a single tradable good that is traded in its domestic market; prior to exchanging goods for currencies in domestic markets, people trade currencies and a linear good on a worldwide market to respond to an aggregate shock with country-specific effects. We consider two setups of domestic markets. The first has competitive markets subject to the CIA constraints. The second has people trading in pairs and the outcome of a pairwise meeting must be in the pairwise core (see Wallace [34]). We use the first setup, one that is close to the Lucas [25] model, to illustrate that the above channel undermines neutrality. We use both setups to examine which regime has an advantage in arranging the future values of currencies to sustain efficient output at each current state.

In the first setup, efficiency is obtained if the rates of return of currencies are stabilized by the Friedman rule (financed by lump sum taxes). When the tax enforcement is through excluding a person from the economy, policy is constrained—the cost of paying current taxes cannot exceed the continuation payoff for the person to stay. In the second setup, we endogenize imperfect substitution of currencies by the scheme of Zhu and Wallace [37] and efficiency is obtained absent of lump sum taxes for patient people by the scheme of Hu et al. [13]. In both setups, the fixed exchange rate is effectively a one-currency regime and the flexible exchange rate splits the joint value of currencies into two country-specific components. In the first setup, given certain stability in the joint value, the split is suboptimal because the variation in each component increases the variation in the stabilization cost; but the flexible exchange-rate regime can be the better regime under other conditions. In the second setup, the split is unambiguously suboptimal because one component can be insufficient for current efficient output in the corresponding country when the joint value is sufficient for the world efficiency.

In the foundation of the OCA literature laid down by Mundell [28], the only benefit of a unified currency is lower transaction costs and the cost is the inability to run country-specific stabilization policy when internal prices are sticky. While the

Friedman rule may be special, the result of the first setup adds to the traditional cost-benefit analysis a seemingly general factor—policy, being country-specific or not, has an endogenous cost to operate. The result of the second setup formalizes the argument of Mundell [29], which, according to McKinnon [27], represents Mundell's argument for the Euro. The essential point of Mundell [29] is that a unified currency better buffers the aggregate shocks as it keeps the general purchasing power of nominal assets stable. Mundell [29] does not consider the influence of the future value of a currency on its current purchasing power; we do.

Below we describe the model in section 2 and analyze the two setups in sections 3 and 4. We discuss more of the related literature and our model in section 5.

2 A two-country variant of the LW model

There are two countries, 1 and 2. Each is populated by a nonatomic unit measure of infinitely-lived people and has its own divisible currency. There are two stages at each discrete date; each stage has a produced and perishable good. The sequence of actions on a date is depicted here:

$$\begin{array}{ccccccc} \text{Individual} & & \text{Three} & & \textbf{Stage 1} & & \textbf{Stage 2} \\ \text{person} & \rightarrow & \text{shocks} & \rightarrow & \text{linear good} & \rightarrow & \text{trade in each} \\ (m_1, m_2) & & \text{realized} & & \text{and two currencies} & & \text{country} \end{array}.$$

That is, each person enters a date with a portfolio (m_1, m_2), where m_k is the amount of currency k (country k's currency) for $k \in \{1, 2\}$. Then, three shocks, two idiosyncratic shocks and one aggregate shock, are realized. One idiosyncratic shock determines a person to be a *producer* or a *consumer* at stage 2 for one date with equal probability; another determines a constant fraction η of consumers in each country to be *tourists* at stage 2 for one date. The aggregate shock determines the current aggregate state; there are I aggregate states; and the transition of states follows a Markov chain with a positive transition matrix (π_{ij}). At stage 1, everyone can produce and consume a linear good (one's utility from consuming y is y and from producing y is $-y$). At stage 2, a producer produces and a nontourist consumer consumes in home while a tourist consumes in foreign country (he returns to home at the end of the date); real international trade consists solely of tourism. When the current state is i, the utility of a consumer who consumes $y \geq 0$ in country k is $u_{ik}(y) \equiv \theta_{ik} u(y)$ and the disutility of a producer who produces $y \geq 0$ in country k is $c_{ik}(y) \equiv \rho_{ik} c(y)$, where $(\theta_{ik}, \rho_{ik}) > (0,0)$, $u(0) = c(0) = 0$, $u' > 0$, $u'' < 0$, $c' > 0$, $c'' \geq 0$, $\beta u'(0) > c'(0)$, and β is the discount factor. We refer to $\alpha = (\theta_{i1}, \rho_{i1}, \theta_{i2}, \rho_{i2})_{i=1}^{I}$ as a *shock vector*. Each person's period utility is the sum of his stage-1 utility and stage-2 utility; he maximizes expected discount utility.

In stage 1, there is a worldwide competitive market for everyone to trade the two currencies and the linear good, where the exchange rate (the relative price of two currencies) is either flexible or fixed. When the exchange rate is fixed, the government of each country is committed to supply unlimited amounts of its own currency, and we normalize the fixed value of the exchange rate as unity. The governments run state-contingent policy $\boldsymbol{\gamma} = (\gamma_{i1}, \gamma_{i2})_{i=1}^{I}$, which withdraws $(1 - \gamma_{ik})s_{ik}$ amount of currency k in the *coming* stage-1 market when i is the *current* state and s_{ik} is the amount of currency k at the end of the current stage 1. A policy is *inactive* if $\gamma_{ik} = 1$ all (i, k) and *active* otherwise. When the policy is active, each person has a country-specific identity for governments to collect *country-specific lump sum* taxes. We renormalize the nominal quantities after policy is implemented so that $s_{i1} = s_{i2} = 0.5$ under the flexible exchange-rate regime (flexible-rate regime hereafter) and $s_{i1} + s_{i2} = 1$ under the fixed rate regime (fixed-rate regime hereafter) at the end of each stage 1. In stage 2, there is a domestic market in each country to trade the stage good.

We limit consideration to stationary equilibria in that when i is the current state, the amount of goods spend on the current stage-1 market to obtain the amount of currency k that is equal to one (renormalized) unit at the end of stage 1 depends only on i. Denote this amount of goods by ϕ_{ik}. The gross expected rate of return from carrying one unit currency k into the coming stage-1 market is

$$\zeta_{ik} = (\gamma_{ik})^{-1} E_i(\phi_{jk}/\phi_{ik}) \tag{1}$$

for the flexible-rate regime (the per unit price of currency k in the coming stage-1 market is $(\gamma_{ik})^{-1}\phi_{jk}$ if j is the next state) and is

$$\zeta_{ik} = (s_{i1}\gamma_{i1} + s_{i2}\gamma_{i2})^{-1} E_i(\phi_{jk}/\phi_{ik}) \tag{2}$$

for the fixed-rate regime (the per unit price of currency k is $(s_{i1}\gamma_{i1} + s_{i2}\gamma_{i2})^{-1}\phi_{jk}$ in the coming stage-1 market if j is the next state), where E_i stands for the expectation made at state i. Denote by $w_{ik}(m_1, m_2)$ the continuation payoff for a country-k resident who leaves the stage-2 market with the portfolio (m_1, m_2). Using a well-known feature of linearity of the stage-1 good, we can express $w_{ik}(m_1, m_2)$ as

$$w_{ik}(m_1, m_2) = \beta(m_1\phi_{i1}\zeta_{i1} + m_2\phi_{i2}\zeta_{i2}) + A_{ik} \tag{3}$$

for some constant A_{ik}; here $m_k\phi_{ik}\zeta_{ik}$ is the (expected and real) future value of the nominal quantity m_k, the key notion indicated in introduction.

A stage-2 allocation or simply an *allocation* is a positive vector consisting of stage-2 output of each country over all states. We focus on stage-2 output because there is no value added in the production of the stage-1 good. An allocation is supported by one regime if it is supported by an equilibrium (i.e., it is an equilibrium allocation) for that regime given some policy; the exchange-rate regime is *neutral* if the two regimes support the same set of allocations and *non-neutral* otherwise. The complete description of the equilibrium conditions depends on the form of the stage-2 domestic markets. We study two different forms in the next two sections.

3 Competitive stage-2 markets

In this section, each stage-2 domestic market is competitive and subject to the country-specific CIA constraint (currency k can only be used in country k). By (3), the rate of return of currency k, ζ_{ik}, cannot exceed $1/\beta$ in any equilibrium. So it is without loss of generality to assume that only consumers who will consume at country k enter stage 2 with currency k. If such a consumer resides in country l and enters stage 2 with m_k units of currency k in state i, then his continuation payoff is

$$W_{ik}(m_k; l) = \max_{(y, m'_k)} u_{ik}(y) + \beta m'_k\phi_{ik}\zeta_{ik} + A_{il}, \tag{4}$$

where $0 \leq y + \upsilon_{ik}m'_k \leq \upsilon_{ik}m_k$ and υ_{ik} is the per unit price of currency k in the stage-2 market; thus his decision problem in the stage-1 market can be expressed as

$$m_{ik} = \arg\max_{m_k \geq 0}[-m_k\phi_{ik} + W_{ik}(m_k; l)]. \tag{5}$$

In (5), m_{ik} does not depend on l because the residency only affects the constant A_{il} in (4). To clear the stage-1 market, we need $m_{i1} = m_{i2} = 1$ for the flexible-rate regime and $m_{i1} + m_{i2} = 2$ for the fixed-rate regime. Let y_{ik} denote the optimal y for the problem (4) with $m_k = m_{ik}$; by the envelope condition, $\phi_{ik} = \upsilon_{ik}u'_{ik}(y_{ik})$. When $\beta\zeta_{ik} < 1$, the CIA constraint $y_{ik} \leq \upsilon_{ik}m_{ik}$ must bind; when $\beta\zeta_{ik} = 1$, it is without loss of generality to limit attention to equilibria with $y_{ik} = \upsilon_{ik} m_{ik}$. It follows that

$$m_{ik}\phi_{ik} = y_{ik} u'_{ik}(y_{ik}). \tag{6}$$

To clear the stage-2 market, a country-k producer must produce y_{ik} given υ_{ik} and w_{ik}, implying $\upsilon_{ik} c'_{ik}(y_{ik}) = \beta\phi_{ik}\zeta_{ik}$ or

$$y_{ik} c'_{ik}(y_{ik}) = \beta m_{ik}\phi_{ik}\zeta_{ik}. \tag{7}$$

Definition 1 *Given policy $\boldsymbol{\gamma}$, a positive price vector $\boldsymbol{\phi} = (\phi_{i1}, \phi_{i2})_{i=1}^{I}$ satisfying (6) and (7) all i is a flexible-rate equilibrium if $m_{i1} = m_{i2} = 1$ all i and is a fixed-rate equilibrium if $\phi_{i1} = \phi_{i2}$ and $m_{i1} + m_{i2} = 2$ all i.*

3.1 An example

For illustration, we work out an example when the policy is inactive. Let $I = 2$, $(\pi_{11}, \pi_{12}) = (0.5, 0.5)$, $\theta_{11} + \theta_{12} = 2$, $\rho_{11} + \rho_{12} = 2$, $|\theta_{11} - 1| \in (0, 1 - \beta)$, $c(y) = 0.5y^2$, and $u(y) = \ln y + L > 0$ if $y \geq \underline{y} > 0$, where L and $\underline{y}$ are constant and $\underline{y}$ is small.[1] Let $(b_{21}, b_{22}) = (b_{12}, b_{11})$ for $b = \theta$, ρ, and π. This symmetry in parameters implies symmetry in equilibrium outcomes, i.e., $(b_{21}, b_{22}) = (b_{12}, b_{11})$ for $b = \phi$, m, and y. Now the consumer's optimal condition (6) is $m_{ik}\phi_{ik} = \theta_{ik}$ and the producer's optimal condition (7) is $\rho_{ik} y_{ik}^2 = \beta m_{ik} E_i \phi_{jk}$.

In the flexible-rate equilibrium, $m_{11} = m_{12} = 1$. So by (6), $(\phi_{11}, \phi_{12}) = (\theta_{11}, \theta_{12})$ and $(E_1\phi_{j1}, E_1\phi_{j2}) = (1, 1)$; then by (7),

$$(y_{11}, y_{12}) = (\sqrt{\beta/\rho_{11}}, \sqrt{\beta/\rho_{12}}). \tag{8}$$

In the fixed-rate equilibrium, $m_{11} + m_{12} = 2$ and $\phi_{11} = \phi_{12}$. So by (6) and $\theta_{11} + \theta_{12} = 2$, we have $\phi_{11} = \phi_{12} = 1$, $(m_{11}, m_{12}) = (\theta_{11}, \theta_{12})$, and $(E_1\phi_{j1}, E_1\phi_{j2}) = (1, 1)$; by (7),

$$(y_{11}, y_{12}) = (\sqrt{\beta\theta_{11}/\rho_{11}}, \sqrt{\beta\theta_{12}/\rho_{12}}). \tag{9}$$

Here the shock shifts demands for currencies solely by its influence on the coefficient θ of the utility function u (a feature of u used in the example); the flexible-rate regime responds with changes in prices ($(\phi_{11}, \phi_{12}) = (\theta_{11}, \theta_{12})$) while the fixed-rate regime with changes in quantities ($(m_{11}, m_{12}) = (\theta_{11}, \theta_{12})$). The future value of one unit of each currency is equal to unity under both regimes so the difference of outputs

[1] For $0 \leq y < \underline{y}$, we can set $u(y) = 2(y/\underline{y})^{0.5} + \ln \underline{y} + L - 2$. All derivations go through if $u(y) = \ln y$ all $y > 0$. The current u satisfies $u(0) = 0$, a property used in Propositions 2 and 3 below.

between (8) and (9) is completely determined by different quantities of currencies generated by the two regimes.

As the exchange-rate regime affects output, it ought to affect the real exchange rate. In this example, as detailed in the appendix, the flexible-rate regime can experience greater volatility of the real exchange rate than the fixed-rate regime. This is qualitatively consistent with the renowned finding of Mussa [30], often viewed as evidence for monetary nonneutrality. By our exercise, Mussa's finding need not invalidate neutrality of money but may support nonneutrality of the exchange-rate regime.

What if the shock does not shift the demands for currencies (even with its influence on θ)? For this to happen, let the two currencies and the linear good be traded in the stage-1 market *before* any shock is realized, and let the two currencies be traded in a *foreign-exchange market* after all shocks are realized but before stage 2 starts; these two markets have the same exchange-rate regime. Now in equilibrium, the price of currency k in the stage-1 market is a constant q_k and each person leaves the market with the portfolio $(0.5, 0.5)$. For the flexible-rate regime, we focus on equilibria that there is no arbitrage gain for producers between the foreign-exchange market and the coming stage-1 market. Then under each regime, a person can by trading on the foreign-exchange market carry the amount of currency k worth of $q = 0.5(q_1 + q_2)$ units of goods in the coming stage-1 market. Hence by a condition analogous to (6), $q = 0.25(1-\eta)(\theta_{11}+\theta_{21})+0.25\eta(\theta_{12}+\theta_{22})+0.5\beta$ (η is the probability for a consumer to be a tourist). So $q = 0.5(1+\beta)$ and neutrality follows from $\rho_{ik}y_{ik}^2 = \beta q$, a condition analogous to (7).[2]

3.2 Results

Our first result is that the exchange-rate regime is nonneutral in a generic sense. Given the transition matrix (π_{ij}), genericity refers to a neighborhood of $1 \in \mathbb{R}^{4I}$ that contains the shock vector α. We first show that there exists a unique flexible-rate equilibrium $\boldsymbol{\phi}(\alpha)$ for the inactive policy when the neighborhood is small (smallness ensures that the rate of return ζ_{ik} does not exceed $1/\beta$ in the candidate equilibrium)

[2] The flexible exchange rate, however, is indeterminate because producers can meet the extra supply or demand of any currency on the part of consumers in the foreign-exchange market when $q_1 + q_2 = 1 + \beta$ and q_1 and q_2 are close to each other. This resembles the finding of King et al. [19] when there are not (intrinsic) aggregate shocks.

and next show that when α is generic, no fixed-rate equilibrium (given any policy) supports the same allocation as the flexible-rate equilibrium $\boldsymbol{\phi}(\alpha)$. For existence, the consumer's optimal condition (6) and the producer's (7) with $m_{ik} = 1$ constitute $4I$ nonlinear equations in unknown $(\boldsymbol{\phi}, \boldsymbol{y})$, where $\boldsymbol{y} = (y_{i1}, y_{i2})_{i=1}^{I}$ is the output vector. As is well known, the $4I$ equations have a unique solution $(\boldsymbol{\phi}^{\circ}, \boldsymbol{y}^{\circ})$ when $\alpha = 1$, where $y_{ik}^{\circ} = y^{\circ}$ and $\phi_{ik}^{\circ} = y^{\circ}u'(y^{\circ})$ all (i, k) and y° satisfies $\beta u'(y^{\circ}) = c'(y^{\circ})$. Then the implicit function theorem implies the unique solution $(\boldsymbol{\phi}(\alpha), \boldsymbol{y}(\alpha))$ for α around 1 (see the online appendix).

Lemma 1 *Given (π_{ij}) and some mild regularity condition, there exist a unique flexible-rate equilibrium $\boldsymbol{\phi}(\alpha)$ for the inactive policy if α is in a neighborhood of $1 \in \mathbb{R}^{4I}$.*

By (6) and (7), $\beta\zeta_{ik}u'_{ik}(y_{ik}) = c'_{ik}(y_{ik})$; that is, stage-2 country-k output in an equilibrium is pinned down by the rate of return of currency k at each state. Because the rates of return of two currencies are equal at each state in any fixed-rate equilibrium, they must be equal in the equilibrium $\boldsymbol{\phi}(\alpha)$ for neutrality to hold. In $\boldsymbol{\phi}(\alpha)$, $\zeta_{ik} = E_i\phi_{jk}/\phi_{ik}$ (see (1)); without loss of generality, let $\phi_{12}/\phi_{11} \geq \phi_{j2}/\phi_{j1}$ all j so

$$\zeta_{11} - \zeta_{12} = \sum_j \pi_{1j}\left(\frac{\phi_{j1}}{\phi_{11}} - \frac{\phi_{j2}}{\phi_{12}}\right) = \sum_j \frac{\pi_{1j}\phi_{j1}}{\phi_{12}}\left(\frac{\phi_{12}}{\phi_{11}} - \frac{\phi_{j2}}{\phi_{j1}}\right) = 0 \qquad (10)$$

holds only if the exchange rate ϕ_{j2}/ϕ_{j1} is constant in j.[3] Thus neutrality forces the flexible-rate equilibrium $\boldsymbol{\phi}(\alpha)$ to fix the exchange rate. As it turns out, the Jacobian matrix of the mapping $\alpha \mapsto (\boldsymbol{\phi}(\alpha), \boldsymbol{y}(\alpha))$ evaluated at $\alpha = 1$ has full rank, meaning that the equilibrium outcome $(\boldsymbol{\phi}(\alpha), \boldsymbol{y}(\alpha))$ should vary with the physical environment α. This seems quite natural. But because fixing the exchange rate limits the freedom of $\boldsymbol{\phi}(\alpha)$ to vary, it can only happen for a measure-zero set of α. Indeed, the dimension of the set $S = \{\alpha : \phi_{i1}(\alpha)\phi_{12}(\alpha) = \phi_{11}(\alpha)\phi_{i2}(\alpha)\}$, viewed as a manifold, is $3I + 1$ (see the online appendix).

Proposition 1 *Given (π_{ij}), the set of α in a neighborhood of $1 \in \mathbb{R}^{4I}$ that permits the two regimes to support the same set of allocations is a measure-zero set.*

A few remarks on Proposition 1 are in order. First, the proposition does not hold if producers do not care about the future values of currencies (e.g., they sell endowed goods); in that case, (7) is not an equilibrium condition so (10) or a fixed exchange

[3] This line of argument is pointed out by Harald Uhlig.

rate is not an implication for the flexible-rate equilibrium when it supports the same allocation as a fixed-rate equilibrium. Second, the measure-zero set in the proposition contains any α with $(\theta_{i1}, \rho_{i1}) = (\theta_{i2}, \rho_{i2})$ all i but it may contain other α (e.g., α with $\theta_{11} = \theta_{12}$ in the example of section 3.1). Lastly, the proposition is adaptable if the number of state-dependent parameters in α increases, if only θ_{ik}s are state-dependent, or if only ρ_{ik}s are state-dependent and $y \mapsto yu'(y)$ is strict monotonic around y°.

Our second result pertains to the optimal regime. We begin with the observation that the flexible-rate regime can imitate the fixed-rate regime to support a same allocation by adopting a suitable active policy. Let $\boldsymbol{\phi}$ be a fixed-rate equilibrium given policy $\boldsymbol{\gamma}$. Let $m'_{ik} = 1$ and let the imitating flexible-rate equilibrium $\boldsymbol{\phi}'$ equate the *current* values of m_{ik} and m'_{ik} in the two regimes, i.e.,

$$m'_{ik}\phi'_{ik} = m_{ik}\phi_{ik}; \tag{11}$$

let the imitating policy $\boldsymbol{\gamma}'$ equate the *future* values of m_{ik} and m'_{ik}, i.e., $\zeta'_{ik}m'_{ik}\phi'_{ik} = \zeta_{ik}m_{ik}\phi_{ik}$ or given (11), $\zeta'_{ik} = \zeta_{ik}$, which by (1) means

$$\gamma'_{ik} = (\phi'_{ik}\zeta_{ik})^{-1}E_i\phi'_{jk}. \tag{12}$$

In fact, (11) are (12) are necessary and sufficient for (6) and (7) to hold when $(m'_{ik}, \phi'_{ik}, \zeta'_{ik})$ is substituted for $(m_{ik}, \phi_{ik}, \zeta_{ik})$. Thus flexibility of choosing currency-specific policy appears to at least allow the flexible-rate regime to be not dominated by the fixed-rate regime. But is it really so?

When taxes are positive in equilibrium, there must be sufficient coercive power for enforcement. Such power is assumed in Definition 1. But consider the scenario that after a person refuses to pay the current taxes, the most severe punishment is to exclude him from all future market activities after the current stage-1 market is over. Then the coercive power is apparently endogenous and there should be a taxation constraint on equilibrium.

To describe the taxation constraint, fix a policy-equilibrium pair $(\boldsymbol{\gamma}, \boldsymbol{\phi})$ and let i be the *current* state and h be the *previous* state. If a country-k resident pays the current taxes τ_{hik}, then his continuation payoff is $-c_{ik}(y_{ik}) + \beta m_{ik}\phi_{ik}\zeta_{ik} + A_{ik}$ as a producer and $u_{il}(y_{il}) - m_{il}\phi_{il} + A_{ik}$ as a consumer to consume in country l; using (6) and (7), $\tau_{hik} \leq U_{ik} + A_{ik}$ is necessary for all country-k residents to pay taxes, where

$$U_{ik} = \min\{-c_{ik}(y_{ik}) + y_{ik}c'_{ik}(y_{ik}), \min_{l\in\{1,2\}}\{u_{il}(y_{il}) - y_{il}u'_{il}(y_{il})\}\}.$$

Hence the total tax revenues $T_{hi} \equiv \sum_k \tau_{hik}$ for the pair $(\boldsymbol{\gamma}, \boldsymbol{\phi})$ are constrained by

$$T_{hi} \leq \sum_k U_{ik} + \sum_k A_{ik}. \tag{13}$$

Referring to (11) and (12), the tax revenues needed by the fixed-rate pair $(\boldsymbol{\gamma}, \boldsymbol{\phi})$ and the flexible-rate pair $(\boldsymbol{\gamma}', \boldsymbol{\phi}')$ there, respectively, to withdraw currencies are

$$T_{hi} = 0.5[\zeta_h \varphi_i \varphi_h (E_h \varphi_j)^{-1} - \varphi_i] \text{ and } T'_{hi} = 0.5[\zeta_h \sum_k \varphi_{ik} \varphi_{hk} (E_h \varphi_{jk})^{-1} - \varphi_i], \tag{14}$$

where $\varphi_{jk} = y_{jk} u'_{jk}(y_{jk})$, $\varphi_j = \sum_k \varphi_{jk}$, and $\zeta_h = \zeta_{h1} = \zeta_{h2}$ in $\boldsymbol{\phi}$; see the appendix for derivation. By definition, $\varphi_{jk} = m_{jk}\phi_{jk} = m'_{jk}\phi'_{jk}$ is the current value of currency k carried into stage 2 at state j for both regimes. Using $E_h b_i = E_h b_j$, we have $E_h T'_{hi} = E_h T_{hi} = 0.5(\zeta_h \varphi_h - E_h \varphi_i)$, i.e., the two regimes demand the same expected tax revenues before the current state i is revealed. After i is realized, the two regimes rely on the same continuation payoff for all people to stay and for each regime, this payoff must cover the different current tax costs stemming from different previous states.

Now apply the *efficient allocation* $(y^*_{i1}, y^*_{i2})_{i=1}^I$ with $u'_{ik}(y^*_{ik}) = c'_{ik}(y^*_{ik})$ all (i,k) to (13) and (14). Given $y_{ik} = y^*_{ik}$, the corresponding policy must be the Friedman rule, which stabilizes the rate of return of each currency at any state at $1/\beta$. Denote by $\pi_{ij}(t)$ the t-step transition probability from i to j and let

$$V_i(\beta) = 2\beta \sum_k U_{ik} + \beta \sum_{t\geq 1} \sum_j \beta^t \pi_{ij}(t) \sum_k [u_{jk}(y^*_{jk}) - c_{jk}(y^*_{jk})]. \tag{15}$$

We can write the taxation constraints (13) for the fixed-rate pair $(\boldsymbol{\gamma}, \boldsymbol{\phi})$ and the flexible-rate pair $(\boldsymbol{\gamma}', \boldsymbol{\phi}')$, respectively, as

$$g_{hi} \equiv \varphi_i \varphi_h (E_h \varphi_j)^{-1} \leq V_i(\beta) \text{ and } f_{hi} \equiv \sum_k \varphi_{ik} \varphi_{hk} (E_h \varphi_{jk})^{-1} \leq V_i(\beta); \tag{16}$$

see the appendix for derivation. Because $V'_i(\beta) > 0$, $V_i(0) = 0$, and $V_i(1) = \infty$, $\beta_{fix}(h,i)$ and $\beta_{flex}(h,i)$ are well defined by

$$g_{hi} = V_i(\beta_{fix}(h,i)) \text{ and } f_{hi} = V_i(\beta_{flex}(h,i)).$$

Let $\beta_{flex} = \max_{(h,i)} \beta_{flex}(h,i)$ and $\beta_{fix} = \max_{(h,i)} \beta_{fix}(h,i)$. Then the efficient allocation is supported by the flexible-rate regime iff $\beta \geq \beta_{flex}$ and by the fixed-rate regime iff $\beta \geq \beta_{fix}$. When the two cutoff values β_{fix} and β_{flex} differ, one regime dominates another over a range of β. To continue, let us focus on the shock vectors α that imply $\varphi_{i1}E_h\varphi_{j2} \neq \varphi_{i2}E_h\varphi_{j1}$ all (h,i), which is a generic property given (π_{ij}). Because

$$f_{hi} - g_{hi} = L_{hi}(\frac{\varphi_{h1}}{\varphi_{h2}} - \frac{E_h\varphi_{j1}}{E_h\varphi_{j2}})(\frac{\varphi_{i1}}{\varphi_{i2}} - \frac{E_h\varphi_{j1}}{E_h\varphi_{j2}}) \tag{17}$$

for some $L_{hi} > 0$, $f_{hi} \neq g_{hi}$ all (h,i).

Suppose either (a) φ_i *is constant in* i or (b) (π_{ij}) *represents an i.i.d process and each* i *has a symmetric state* $\sigma(i)$*, i.e., there is a mapping* σ *on* $\{1,...,I\}$ *such that* $\sigma(\sigma(i)) = i$ *and* $(\alpha_{\sigma(i)1}, \alpha_{\sigma(i)2}) = (\alpha_{i2}, \alpha_{i1})$ *all* i *(*$\alpha_{ik} = (\theta_{ik}, \rho_{ik})$*),* $\pi_{ij} = \pi_{\sigma(i)\sigma(j)}$ *all* (i,j)*, and* $\sigma(i)) = i$ *at most one* i. Then given any current state i, for any previous state h, there is a previous state h' satisfying $f_{h'i} > g_{hi}$, i.e., the flexible-rate regime incurs a higher tax spike than the fixed-rate regime at i. Indeed, when condition (a) holds, $g_{hi} = g_{ii}$; by (17), $f_{ii} > g_{ii}$ $(h' = i)$. When condition (b) holds, $g_{\sigma(h)i} = g_{hi}$, $(\varphi_{h1}, \varphi_{h2}) = (\varphi_{\sigma(h)2}, \varphi_{\sigma(h)1})$, and the value of $E_h\varphi_{jk}$ does not depend on (h,k); then by (17), $\max\{f_{hi}, f_{\sigma(h)i}\} > g_{hi}$ $(h' = h$ or $\sigma(h))$. Because i is arbitrary, $\beta_{flex} > \beta_{fix}$.

This result is intuitive. The fixed-rate regime has a constant cost at the current state to stabilizing the rates of return of currencies over all (previous) states or over each pair of symmetric states because the joint value φ_j of two currencies is constant over all states or over the pair. Then, splitting φ_j into two country-specific components φ_{1j} and φ_{2j} by the flexible-rate regime is suboptimal as the variation in each component leads to the variation in the stabilization cost. Note that $\beta_{flex} > \beta_{fix}$ holds for $\tilde{\alpha}$ in a neighborhood of α that satisfies condition (a) and for $(\tilde{\alpha}, (\tilde{\pi}_{ij}))$ in a neighborhood of $(\alpha, (\pi_{ij}))$ that satisfies condition (b). In summary, certain sort of stability in the joint value of currencies carried into stage 2 gives rise to the dominance of the fixed-rate regime.

We find no conditions that are easily described and verified (as conditions (a) and (b)) to ensure $\beta_{flex} < \beta_{fix}$. An obvious direction is that $f_{ih} < g_{ih}$ for some (i,h) that attains β_{flex}. We provide such an example in the appendix. Among others, the shock exerts much asymmetric effects on two countries in that example.

Proposition 2 *In the presence of the taxation constraint, given* (π_{ij})*, the fixed-rate regime has a lower cutoff value for* β *to sustain the efficient allocation than the flexible-rate regime as long as* α *implies* $y^*_{ik} > 0$ *all* (i,k) *and a sufficiently small cross-state variation in* $\sum_k y^*_{ik}u'_{ik}(y^*_{ik})$ *and is outside a measure-zero set in* $\mathbb{R}^{4I}$*. In general, which regime has a lower cutoff value may depend on the aggregate shock and preferences.*

4 Decentralized stage-2 markets

In this section, each consumer randomly meets a producer in each stage-2 domestic market. Critical for this setup is which trade is selected by agents in a pairwise meeting. Following Wallace [34], we require that a selection be in the pairwise core. With this approach, as shown below, we can endogenize imperfect substitution of currencies. So we do not appeal to CIA constraints to eliminate the Kareken-Wallace [18] indeterminacy. Moreover, also as shown below, efficiency can be attained absent of lump sum taxes for patient people. So we only consider the inactive policy, which may be justified by the assumption that people are all anonymous.[4]

For a country-k meeting in state i, we denote by (y, κ, ι) a generic trading outcome, where y is the producer's output, κ is the consumer's payment the producer's home currency (i.e., currency k), and ι is the payment in the producer's foreign currency. When a consumer carries $m = (m_1, m_2)$ and a producer carries $m' = (m'_1, m'_2)$, we denote by $(y_{ik}(n), \kappa_{ik}(n), \iota_{ik}(n))$ the outcome selected from the pairwise core. The pairwise core is determined by $n = (m, m')$, u_{ik}, c_{ik}, and the continuation payoffs w_{i1} and w_{i2} in (3) (now $\zeta_{ik} = E_i\phi_{jk}/\phi_{ik}$ in (3) as the policy is inactive); the selected outcome is the solution to a two-step problem adapted from Zhu and Wallace [37].

Problem 1 *Fix k, i and $n = (m, m')$. Let $l \in \{1, 2\}$ and $l \neq k$. Define $\bar{m}$ by $\bar{m}_k = m_k$ and $\bar{m}_l = 0$.*

Step 1. Select some trading outcome $(\bar{y}_{ik}(n), \bar{\kappa}_{ik}(n), 0)$ from the pairwise core when the producer carries m' and the buyer carries $\bar{m}$.

Step 2. Using step-1 input $(\bar{y}_{ik}(n), \bar{\kappa}_{ik}(n))$, let

$$(y_{ik}(n), \kappa_{ik}(n), \iota_{ik}(n)) = \arg\max_{(y,\kappa,\iota)}[-c_{ik}(y) + \beta E_i(\kappa\phi_{jk} + \iota\phi_{jl})] \tag{18}$$

subject to $0 \leq \kappa \leq m_k$, $0 \leq \iota \leq m_l$, and

$$u_{ik}(y) - \beta E_i(\kappa\phi_{jk} + \iota\phi_{jl}) \geq u_{ik}(\bar{y}_{ik}(n)) - \beta E_i\bar{\kappa}_{ik}(n)\phi_{jk}. \tag{19}$$

In Problem 1, the step-1 selection effectively turns the producer's home currency as the favored asset in the meeting. The outcome selected by (18) maximizes the

[4] An alternative approach is to adopt a generalized Nash bargaining solution as in Lagos and Wright [24], impose CIA constraints, and also consider active policies. Then results are comparable to those in section 3. But the meeting trading outcome can be no in the pairwise core as the trading outcome in the section-3 setup is not in the core defined for all domestic-market participants.

producer's payoff conditional on not making the consumer worse off than the step-1 selection $(\bar{y}_{ik}(n), \bar{\kappa}_{ik}(n), 0)$; it is in the pairwise core (when the consumer carries m and the producer carries m') because there is no restriction on which asset can be used in payments.[5] Problem 1 does not represent an extensive game form with two rounds of alternating offers but it may be understood as a gradual bargaining problem (see O'Neill et al. [31]). Hu et al. [13] implement the solution to such a problem by a simple game form used in Zhu [36].

As in section 3, the rate of return ζ_{ik} cannot not exceed $1/\beta$ in any equilibrium. Because the consumer in Problem 1 is indifferent between $(y_{ik}(n), \kappa_{ik}(n), \iota_{ik}(n))$ and $(\bar{y}_{ik}(n), \bar{\kappa}_{ik}(n), 0)$ (the constraint (19) must bind), $\beta\zeta_{ik} \leq 1$ implies no strict benefit for him to carry currency $l \neq k$ into the meeting. Thus it is without loss of generality to assume that only consumers who will consume in country k enter stage 2 with currency k and concentrate on the problem of such a consumer in the stage-1 market,

$$m_{ik} = \arg\max_{m_k \geq 0} [-m_k \phi_{ik} + u_{ik}(\bar{y}_{ik}(n)) + \beta(m_k - \bar{\kappa}_{ik}(n)) E_i \phi_{jk}] \tag{20}$$

for n with $m' = (0, 0)$ and m with $m_l = 0$ $(l \neq k)$.

Definition 2 *Given the step-1 selection in Problem 1, a positive price vector* $\boldsymbol{\phi} = (\phi_{i1}, \phi_{i2})_{i=1}^{I}$ *is a flexible-rate equilibrium if* $m_{i1} = m_{i2} = 1$ *all* i *and is a fixed-rate equilibrium if* $\phi_{i1} = \phi_{i2}$ *and* $m_{i1} + m_{i2} = 2$ *all* i.

Following the argument in Hu et al. [13], one can translate a Definition-2 equilibrium into an equilibrium of a social planner's mechanism-design problem.

Thus far we have not specified a concrete form for the step-1 selection in Problem 1. There are many alternatives. For example, the selection may assign all surplus to the consumer conditional on the consumer only carrying the favored asset into the meeting; with this selection, it is straightforward to adapt the argument for Proposition 1 to show that the two regimes are different. Our focus is a selection that supports the efficient allocation $(y_{i1}^*, y_{i2}^*)_{i=1}^{I}$ $(u'_{ik}(y_{ik}^*) = c'_{ik}(y_{ik}^*)$ all $(i, k))$ when people are patient.

[5] The Zhu-Wallace scheme is used in Nosal and Rocheteau [32] to obtain a determinate exchange rate under flexible exchange rates. Such determinacy is obtained by Head and Shi [12] in a large-family model when each family member can hold only one sort of currency to search. Imperfect substitution may be a consequence of intrinsic difference between currencies; for example, some currency is harder to counterfeit (see Gomis-Porqueras et al. [11] and Zhang [35] for related models). No intrinsic difference is a useful reference when one views the exchange-rate regime as a policy choice; recall that the fixed-rate regime is effectively a unified-currency regime in our model.

We begin with necessary conditions to support the efficient allocation. Let $\boldsymbol{\phi}$ be a supporting equilibrium. Consider a country-k meeting in state i where the consumer carries m with $m_k = m_{ik}$ and $m_l = 0$ $(l \neq k)$ and the producer carries $m' = (0, 0)$; let $\kappa \leq m_{ik}$ be the payment of currency k in the meeting. A necessary condition for the producer to produce y^*_{ik} is

$$c_{ik}(y^*_{ik}) \leq \beta\kappa E_i \phi_{jk}; \tag{21}$$

a necessary condition for the consumer to carry m_{ik} into the meeting is

$$m_{ik}\phi_{ik} \leq u_{ik}(y^*_{ik}) + \beta(m_{ik} - \kappa)E_i\phi_{jk}. \tag{22}$$

When $\boldsymbol{\phi}$ is a flexible-rate equilibrium, $m_{ik} = 1$, (21) and (22) imply

$$\phi_{ik} \leq u_{ik}(y^*_{ik}) - c_{ik}(y^*_{ik}) + \beta E_i \phi_{jk};$$

that is, the cost for the consumer to carry one unit of currency k into the meeting cannot exceed the consumer-producer joint trading surplus together with the discount future value of this unit. By a simple fixed-point argument, there exists a unique vector $(x_{i1}, x_{i2})_{i=1}^I$ such that x_{ik} is the maximal possible current value of currency k in state i (so $x_{ik} \geq \phi_{ik}$) and

$$x_{ik} = u_{ik}(y^*_{ik}) - c_{ik}(y^*_{ik}) + \beta E_i x_{jk} \tag{23}$$

all (i, k).[6] By repeated substitution, (23) yields the maximal possible discount future value of currency k,

$$\beta E_i x_{jk} = \sum_{t \geq 1} \sum_j \beta^t \pi_{ij}(t)[u_{ik}(y^*_{ik}) - c_{ik}(y^*_{ik})] \equiv v_{ik}(\beta), \tag{24}$$

where $\pi_{ij}(t)$ is the t-step transition probability from state i to j as in (15). When $\boldsymbol{\phi}$ is a fixed-rate equilibrium, $\sum_k m_{ik} = 2$, (21), and (22) imply

$$2\phi_i \leq \sum_k [u_{ik}(y^*_{ik}) - c_{ik}(y^*_{ik})] + 2\beta E_i \phi_j,$$

where $\phi_j = \phi_{j1} = \phi_{j2}$; that is, the total cost for a consumer to consume in country 1 and a consumer to consume in country 2 to carry totally two units of currencies cannot exceed the joint surplus over the two relevant meetings together with the

[6] Let $\bar{\boldsymbol{\phi}} \in \mathbb{R}^{2I}_+$ have the constant element equal to $(1 - \beta)^{-1} \max_{(i,k)}[u_{ik}(y^*_{ik}) - c_{ik}(y^*_{ik})]$. Define $x \mapsto H(x)$ from $\{x : \bar{\boldsymbol{\phi}} \geq x \geq \boldsymbol{\phi}\} \subset \mathbb{R}^{2I}_+$ to $\mathbb{R}^{2I}$ by $H_{ik}(x) = u_{ik}(y^*_{ik}) - c_{ik}(y^*_{ik}) + \beta E_i x_{jk}$. Observe that H is a contraction mapping and $H(\bar{\boldsymbol{\phi}}) \geq H(x) \geq H(\boldsymbol{\phi})$.

discount future value of these units. Now there exists a unique vector $(z_i)_{i=1}^I$ such that z_i the maximal possible joint current value of currencies in state i (so $z_i \geq 2\phi_i$) and

$$z_i = \sum_k [u_{ik}(y_{ik}^*) - c_{ik}(y_{ik}^*)] + \beta E_i z_j \tag{25}$$

all i, which yields the maximal possible discount joint future value

$$\beta E_i z_j = \sum_{t \geq 1} \sum_j \beta^t \pi_{ij}(t) \sum_k [u_{ik}(y_{ik}^*) - c_{ik}(y_{ik}^*)] \equiv v_i(\beta). \tag{26}$$

Note $v_{ik}'(\beta), v_i'(\beta) > 0$, $v_{ik}(0) = v_i(0) = 0$, and $v_{ik}(1) = v_i(1) = \infty$. So $\beta(i,k)$ and $\beta(i)$ are well defined by

$$v_{ik}(\beta(i,k)) = c_{ik}(y_{ik}^*) \text{ and } v_i(\beta(i)) = \sum_k c_{ik}(y_{ik}^*). \tag{27}$$

Let $\beta_{flex} = \max_{(i,k)} \beta(i,k)$ and $\beta_{fix} = \max_i \beta(i)$. Because the discount future value $\beta E_i \phi_{jk}$ of currency k is bounded above by $v_{ik}(\beta)$, the producer will not produce y_{ik}^* ((21) cannot hold) under the flexible-rate regime if $\beta < \beta(i,k)$. Because the discount joint future value $2\beta E_i \phi_j$ of currencies is bounded above by $v_i(\beta)$, the producer will not produce y_{ik}^* under the fixed-rate regime for either $k = 1$ or $k = 2$ if $\beta < \beta(i)$. Hence $\beta \geq \beta_{flex}$ is necessary for the flexible-rate regime to support the efficient allocation and $\beta \geq \beta_{fix}$ is necessary for the fixed-rate regime.

Next we show that these necessary conditions are sufficient. To this end, we borrow the scheme from Hu et al. [13] and Hu and Rocheteau [14] to determine the step-1 selection in Problem 1. Such a scheme is characterized by a list $\{m_{ik}^*\}$. For a country-k meeting in i, if the consumer carries at least m_{ik}^* amount of currency k, then he receives all of the surplus conditional on he only carrying currency k; otherwise, the producer receives all of the surplus. That is, if $m_k \geq m_{ik}^*$ in n then

$$(\bar{y}_{ik}(n), \bar{\kappa}_{ik}(n)) = \arg\max_{(y,\kappa)} [u_{ik}(y) - \beta\kappa E_i \phi_{jk}] \tag{28}$$

subject to $0 \leq \kappa \leq m_k$ and $-c_{ik}(y) + \beta\kappa E_i \phi_{jk} \geq 0$; otherwise,

$$(\bar{y}_{ik}(n), \bar{\kappa}_{ik}(n)) = \arg\max_{(y,\kappa)} [-c_{ik}(y_{ik}) + \beta\kappa_{ik} E_i \phi_{jk}] \tag{29}$$

subject to $0 \leq \kappa \leq m_k$ and $u_{ik}(y) - \beta\kappa E_i \phi_{jk} \geq 0$. The feature of the scheme in (28) and (29) is that the consumer's payoff from the trade as a function of m_k is not smooth, actually not continuous, at m_{ik}^*; such nonsmoothness is the key to incentivize the consumer to carry m^*_{ik} amount of currency k into the meeting when currency k is sufficiently valued (a sufficient value at each state gives rise to a sufficient future value) but the rate of return of holding k is low (i.e, when $\zeta_{ik} < 1/\beta$).

Now for the flexible-rate regime, let $m^*_{ik} = 1$; if $\beta \geq \beta_{flex}$, then the maximal price vector $\boldsymbol{\phi}$ defined by $\phi_{ik} = x_{ik}$ all (i,k) (x_{ik} is given by (23)) is a flexible-rate equilibrium. For the fixed-rate regime, let $m^*_{ik} = 2c_{ik}(y^*_{ik})[c_{i1}(y^*_{i1}) + c_{i2}(y^*_{i2})]^{-1}$; if $\beta \geq \beta_{fix}$, then the maximal price vector $\boldsymbol{\phi}$ defined by $\phi_{ik} = 0.5z_i$ all (i,k) (z_i is given by (25)) is a fixed-rate equilibrium. To verify, consider the flexible rate and the argument the fixed rate is similar. By definition, $\beta\zeta_{ik} < 1$ (see (23)) and $\beta E_i\phi_{jk} = v_{ik}(\beta)$ (see (24)). Fix a country-k meeting in state i between a consumer who resides in country k' and carries m with $m_l = 0$ $(l \neq k)$ and a producer who carries $m' = (0,0)$. Let y denote the output and κ the payment of currency k selected by the scheme in (28) and (29). If $m_k \geq 1$, then by (28), $c_{ik}(y) = \kappa v_{ik}(\beta)$ and $u'_{ik}(y) \geq c'_{ik}(y)$, strict only if $c_{ik}(y^*_{ik}) > m_k v_{ik}(\beta)$. Given $\beta \geq \beta_{flex}$, $v_{ik}(\beta) \geq c_{ik}(y^*_{ik})$ so $y = y^*_{ik}$ and the consumer's payoff is $u_{ik}(y^*_{ik}) - c_{ik}(y^*_{ik}) + m_k v_{ik}(\beta) + A_{ik'}$. If $m_k < 1$ then by (29), the consumer's payoff is $m_k v_{ik}(\beta) + A_{ik'}$. Because the utility cost to carrying m_k into the meeting is $m_k\phi_{ik}$ and $\zeta_{ik} < 1/\beta$, it is optimal for the consumer to leave the stage-1 market with $m_k = 1$.

In summary, the efficient allocation is supported by the flexible-rate regime iff $\beta \geq \beta_{flex}$ and by the fixed-rate regime iff $\beta \geq \beta_{fix}$. To compare the two cutoff values, suppose $\beta(i) \geq \hat{\beta}_i \equiv \max\{\beta(i,1), \beta(i,2)\}$. Then

$$\sum_k c_{ik}(y^*_{ik}) = v_i(\beta(i)) \geq v_i(\hat{\beta}_i) = \sum_k v_{ik}(\hat{\beta}_i) \geq \sum_k v_{ik}(\beta(i,k)) = \sum_k c_{ik}(y^*_{ik}). \quad (30)$$

In (30), the equalities use definitions of $\beta(i)$, $v_{ik}(\beta)$, $v_i(\beta)$, and $\beta(i,k)$ (see (27), (24), and (26)); the first inequality uses $v'_i(\beta) > 0$ and it is strict if $\beta(i) > \hat{\beta}_i$, and the second uses $v'_{ik}(\beta) > 0$ and it is strict if $\beta(i,1) \neq \beta(i,2)$. So $\beta(i) \leq \hat{\beta}_i$ and strict if $\beta(i,1) \neq \beta(i,2)$. Because $\beta(i,1) = \beta(i,2)$ is not generic, we reach the following.

Proposition 3 *Given (π_{ij}), the fixed-rate regime has a lower cutoff value for β than the flexible-rate regime to support the efficient allocation as long as α implies $y^*_{ik} > 0$ all (i,k) and is not in a measure-zero set in $\mathbb{R}^4$.*

The logic behind Proposition 3 may be iterated as follows. Given the current state i, the flexible-rate regime requires that the discount future value of one unit of currency k reach $v_{ik}(\beta)$ to cover the utility cost $c_{ik}(y^*_{ik})$ from producing y^*_{ik} in country k; the fixed-rate regime requires that the joint discount future value of two units of currencies reach $v_i(\beta)$ to cover $\sum_k c_{ik}(y^*_{ik})$ from producing $\sum_k y^*_{ik}$ worldwide. Splitting the joint value $v_i(\beta)$ into the two country-specific components $v_{i1}(\beta)$ and $v_{i2}(\beta)$ brings no strict gain but creates a strict loss when $v_{ik}(\beta)$ falls below $c_{ik}(y^*_{ik})$. This logic is generalizable when imperfect substitution of currencies is endogenized by a scheme other than the one used in Problem 1 but obeying the pairwise-core restriction. If such a scheme induces no currency substitution at all (as Problem 1), then Proposition 3 carries over; otherwise, we have a weaker conclusion in general—the flexible-rate regime cannot beat the fixed-rate regime because $v_i(\beta)$ remains to be the upper bound on the discount joint future value of currencies which can be attained by the fixed-rate regime and the split of which cannot yield strict gain.

It may help to relate supporting efficiency here to supporting efficiency in the section-3 setup. In both setups, the flexible-rate regime splits the joint value of two currencies (in the current term and in the future term) into two country-specific components. In the section-3 setup, people pay taxes today to finance policy that stabilizes yesterday's rates of return of currencies at $1/\beta$ for yesterday's efficient output. Although the split made by the flexible-rate regime does not affect the benefits for people to pay current taxes, it affects the current tax payments. Here, the scheme in (28) and (29) incentivizes consumers to not economize holdings of currencies when currencies at each state are sufficiently valued but have rates of return lower than $1/\beta$. The split made by the flexible-rate regime affects the usage of the joint future value for today's production. When the taxation constraint is not binding for any regime in the section-3 setup or when the future value of each currency is abundant here (i.e., when $\beta \geq \max\{\beta_{fix}, \beta_{flex}\}$), different quantities generated by the two regimes can support the same efficient output in the section-3 setup because policy equates the future values of different quantities (see (12)) and here because consumers do not spend all currencies in hand.

Ultimately, the difference between the two setups comes down to the fact that decentralized trade admits a non-degenerate core for the parties who trade with each other, allowing the endogenous equilibrium arrangements that favor one asset over another and signify a specific amount of a given asset. Those arrangements break down the ties between the current output and the rates of return of assets, ties that call for the CIA constraints to generate imperfect substitution and for the Friedman rule to sustain sufficient future real values of currencies in the section-3 setup.

5 Discussion

A group of papers concern whether one or two currencies better facilitate bilateral trades. Matsuyama et al. [26] make the point that when there is no intrinsic difference between currencies, restrictions on which currency may be used in certain trades do not help. Kocherlakota and Krueger [22] exploit the signaling advantage of two currencies in the presence of asymmetric information. Kocherlakota [21] and Dong and Jiang [7] emphasize the gain of two currencies in expanding the recording-keeping dimension. Kiyotaki and Moore (2002) find that one currency may hurt specialization. Araujo and Ferrais [2] show that two currencies help shift misallocated liquidity. All these papers abstract away aggregate shocks.

In the traditional OCA literature, Mundell [29] is the exceptional effort to analyze benefits of a unified currency from the monetary aspect. Motivated by Mundell [29], Ching and Devereux [5] study a one-period MUF model in which two currencies prohibit the international risk sharing because only home currency enters into one's utility function and one currency, which is always home currency, admits. Also connecting the traditional OCA literature from the monetary aspect, Cooper and Kempef [6] formulate the benefit and cost of a unified currency in Mundell [28] by an OLG model with flexible prices and aggregate shocks. They capture the transaction cost of multiple currencies by an infinite cost for old people to adjust portfolios and the policy inflexibility of a unified currency by the equal sharing rule of inflation taxes between two countries.

Cooper and Kempef [6] do not study the fixed exchange rate. Because old people cannot adjust portfolios, the fixed exchange rate renders the quantities of the two currencies at the end of the last period as an endogenous state variable for the current period (if old can, then the distribution of portfolios of olds is a state variable for both regimes, making the model hard to analyze). To the best of our knowledge, Lahiri et al. [16] are unique in the literature in explicitly claiming nonneutrality under flexible prices. Adapting the CIA model of Alvarez et al. [1], Lahiri et al. [16] show that with market segmentation, two exchange-rate regimes lead to different wealth redistributions in a small open economy. As Alvarez et al. [1], Lahiri et al. [16] study an endowment economy.

Our paper explores a simple mechanism—people care about the future values of their current nominal earnings. We show that this mechanism is sufficient to invalidate neutrality and we add to the OCA literature what may be a better regime to arrange the future values of currencies. Needless to say, our model is special in many aspects. For example, there is no direct substitutability between home and foreign goods and all trade is spot *quid pro quo* trade. We may let the tourist status be endogenous and some trade to be not spot trade for currencies (see Araujo and Hu [3] for an analysis of credit in the LW model). In any case, we suspect that the basic insights carry over as long as some producers must be paid by currencies.

Appendix

The real-exchange rate in section-3.1 example

The real exchange rate at state i is determined as

$$rer_i = \frac{\phi_{i1}}{\phi_{i2}} \frac{(1-\nu_{i1})(1/\phi_{i1}) + \nu_{i1}(1/y_{i1})}{(1-\nu_{i2})(1/\phi_{i2}) + \nu_{i2}(1/y_{i2})}. \tag{31}$$

In (31), $(1-\nu_{ik})(1/\phi_{ik})+\nu_{ik}(1/y_{ik})$ measures the *country-k price level* and $1-\nu_{ik}$ and ν_{ik} are weights for the price of the stage-1 good in the unit of currency k and the price of the stage-2 good produced in country k in the unit of currency k, respectively. The weight ν_{ik} is determined by the contribution of the stage-2 domestic output to total country-k output. Because there is no value added in the production of the stage-1 good, net output at stage 1 is zero and, hence, $\nu_{ik} = 1$. Applying this to (31) and using (6), we have $rer_i = u'_{i1}(y_{i1})/u'_{i2}(y_{i2})$.

As it turns out, persistence of the shock is a factor for the real exchange rate in the flexible-rate equilibrium. So let $(\pi_{11}, \pi_{12}) = (\mu, 1-\mu)$ and $\mu \geq 0.5$. This generalization does not affect any data for the fixed-rate equilibrium given in section 3.1. In the flexible-rate equilibrium, $(E_1\phi_{j1}, E_1\phi_{j2}) = (\mu\theta_{11} + (1-\mu)\theta_{12}, \mu\theta_{12} + (1-\mu)\theta_{11})$ and $(y_{11}, y_{12}) = (\sqrt{\beta E_1\phi_{j1}/\rho_{11}}, \sqrt{\beta E_1\phi_{j2}/\rho_{12}})$. Let $\delta_\theta = \theta_{11} - 1$ and $\delta_\rho = \rho_{11} - 1$. By a first order approximation, the variances of output and the real exchange rate in each country are $0.25[(1-2\mu)\delta_\theta + \delta_\rho]^2$ and $[(1+2\mu)\delta_\theta + \delta_\rho]^2$, respectively, in the flexible-rate equilibrium; they are $0.25(\delta_\theta + \delta_\rho)^2$ and $(2\delta_\theta + \delta_\rho)^2$ in the fixed-rate equilibrium. In each equilibrium, the autocorrelations of output and the real exchange rate in each country are $2\mu - 1$.

Derivation of (14) and (16)

For (14), let $\phi_j = \phi_{j1} = \phi_{j2}$. First consider T_{hi}. The per-unit price of each currency is ϕ_i/a_h on the current market and $1 - a_h$ units of currencies are withdrawn, where $a_h = s_{h1}\gamma_{h1} + s_{h2}\gamma_{h2}$ (see (2)). So $T_{hi} = (\phi_i/a_h)(1 - a_h)$. By $a_h\zeta_h\phi_h = E_h\phi_j$, $T_{hi} = \varphi_i(\zeta_h\phi_h/E_h\phi_j - 1)$ and the expression in (14) follows from $m_{ik}\phi_i = m_{ik}\phi_{ik} = \varphi_{ik}$ (see (6)), $m_{i1} + m_{i2} = 2$, and $\varphi_{i1} + \varphi_{i2} = \varphi_i$. Next consider T'_{hi}. The per-unit price of currency k is ϕ'_{ik}/γ'_{hk} and $0.5(1 - \gamma'_{hk})$ units of currency k are withdrawn. So $T'_{hi} = 0.5\sum_k(\phi'_{ik}/\gamma'_{hk})(1 - \gamma'_{hk})$. By (12), $T'_{hi} = 0.5\sum_k[\phi'_{ik}\phi'_{hk}\zeta_h/E_h\phi'_{jk} - \phi'_{ik}]$ and by (11), $T'_{hi} = 0.5\zeta_h\sum_k \phi_i m_{ik}\phi_h m_{hk}/E_h m_{jk}\phi_j - \phi_i$. Then the expression in (14) follows from $m_{ik}\phi_i = m_{ik}\phi_{ik} = \varphi_{ik}$. For (16), we have $A_{ik} = 0.5\beta E_i[-2\tau_{ijk} + \Delta_{jk} + 2A_{jk} - \Lambda_{jk}]$, where $\Delta_{jk} = (1-\eta)u_{jk}(y^*_{jk}) + \eta u_{jl}(y^*_{jl}) - c_{jk}(y^*_{jk})$ (η is the probability for a consumer to be a tourist) and $\Lambda_{jk} = \eta y^*_{jk}u'_{jk}(y^*_{jk}) + (1-\eta)y^*_{jl}u'_{jl}(y^*_{jl}) - y^*_{jk}c'_{jk}(y^*_{jk})$ with $l \neq k$. Let $\Delta_j = \sum_k[u'_{jk}(y^*_{jk}) - c'_{jk}(y^*_{jk})]$. Using $E_iT_{ij} = 0.5\beta^{-1}\varphi_i - 0.5E_i\varphi_j$ and $u'_{jk}(y^*_{jk}) = c'_{jk}(y^*_{jk})$, then $A_i \equiv A_{i1} + A_{i2} = -0.5\varphi_i + 0.5\beta E_i(\Delta_j + \varphi_j + 2A_j)$. By repeated substitution, $A_i = -0.5\varphi_i + 0.5\sum_{t\geq 1}\sum_j \beta^t\pi_{ij}(t)\Delta_j$. Now (16) follows from applying this A_i to (14) and $\zeta_h = \beta^{-1}$ to (13).

Example of $\beta_{flex} < \beta_{fix}$ in Proposition 2

Let $c(y) = y$ and let u be the same as in the example in section 3.1. Let $I = 3$. Let $(\pi_{i1}, \pi_{i2}, \pi_{i3}) = (\mu_1\lambda_1, \mu_1\lambda_2, \mu_2)$ for $i = 1, 2$ and $(\pi_{i1}, \pi_{i2}, \pi_{i3}) = (\mu_2/2, \mu_2/2, \mu_1)$, where $\lambda_1 + \lambda_2 = \mu_1 + \mu_2 = 1$. Let (i) $\theta_{31} = \theta_{11} > \theta_{21}$, (ii) $\theta_{32} > \theta_{12} > \theta_{22}$, and (iii) $\theta_{11}/\theta_{12} > E_1\theta_{j1}/E_1\theta_{j2} > \theta_{31}/\theta_{32}$. By (iii) and (17), $g_{13} > f_{13}$. A simple way to ensure that $(1, 3)$ attains β_{flex} is to vary μ_2 and ρ_{ik}. To see how this work, first note that given (i) and (ii), $f_{13} \geq f_{hi}$ for $h, i \in \{1, 2\}$ (note $E_1\theta_{ik} = E_2\theta_{ik}$). Using (i) and (ii) once more, we have $(1, 3) = \text{argmax} f_{hi}$ if $f_{13} > f_{33}$. When μ_2 is close to 0, $\theta_{1k}/E_1\theta_{ik} > \theta_{3k}/E_3\theta_{ik}$ so (i) and (ii) ensure $f_{13} > f_{33}$. Note that linearity of c implies $A_{ik} = 0$ so by adjusting ρ_{ik}, we can further ensure$V_1(\beta) = V_2(\beta) > V_3(\beta)$.

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Online appendix

Proof of Lemma 1

We assume the following regularity condition:

Condition 1 *$D_0 \neq \beta\pi_{ii}D_1$ all i, where $D_0 = c'(y^\circ) + y^\circ c''(y^\circ)$ and $D_1 = u'(y^\circ) + y^\circ u''(y^\circ)$.*

The $4I$ equations in the main text can be written as $F_{ik}(\boldsymbol{\phi}, \boldsymbol{y}, \alpha) = 0$ all (i,k), where $F_{ik} = (F^c_{ik}, F^p_{ik})$ and

$$F^c_{ik}(\boldsymbol{\phi}, \boldsymbol{y}, \alpha) = \phi_{ik} - y_{ik}\theta_{ik}u'(y_{ik}), \tag{32}$$

$$F^p_{ik}(\boldsymbol{\phi}, \boldsymbol{y}, \alpha) = y_{ik}\rho_{ik}c'(y_{ik}) - \beta E_i\phi_{jk}; \tag{33}$$

Now $\partial F^c_{ik}/\partial\phi_{ik} = 1$; $\partial F^p_{ik}/\partial\phi_{jk} = -\beta\pi_{ij}$ (all j), $\partial F^c_{ik}/\partial y_{ik} = -\theta_{ik}[u'(y_{ik}) + y_{ik}u''(y_{ik})]$, and $\partial F^p_{ik}/\partial y_{ik} = \rho_{ik}[c'(y_{ik}) + y_{ik}c''(y_{ik})]$; $\partial F^c_{ik}/\partial\phi_{jk}$, $\partial F^c_{ik}/\partial y_{jk}$, and $\partial F^p_{ik}/\partial y_{jk}$ vanishes if $j \neq i$. Hence, the Jacobian matrix of $(F_{1k}, ..., F_{Ik})$ with respect to $(\boldsymbol{\phi}, \boldsymbol{y})$ evaluated at $(\boldsymbol{\phi}, \boldsymbol{y}, \alpha) = (\boldsymbol{\phi}^\circ, \boldsymbol{y}^\circ, 1)$ is

$$\partial F_{\phi yk} = \begin{bmatrix} \mathbf{I} & \vdots & -D_1\mathbf{I} \\ \cdots & \cdots & \cdots \\ -\beta\mathbf{\Pi} & \vdots & D_0\mathbf{I} \end{bmatrix}, \tag{34}$$

where $\mathbf{\Pi} = (\pi_{ij})$ and $\mathbf{I}$ is the $I \times I$ identity matrix. By its structure, the matrix in (34) is invertible if its ith and $(i+I)$th columns are linearly independent for $1 \leq i \leq I$. This is the case if the ith and $(i+1)$th rows of these two columns constitute an invertible matrix. That, in turn, follows from Condition 1. Then by the implicit function theorem, there exists a unique $(\boldsymbol{\phi}(\alpha), \boldsymbol{y}(\alpha))$ for α in a neighborhood N of $1 \in \mathbb{R}^4$. That $\boldsymbol{\phi}(\alpha)$ is unique is implied by (33).

Proof of Proposition 1

To begin with, let the mapping Φ on the neighborhood N of $1 \in \mathbb{R}^{4I}$ be defined by $\Phi(\alpha) = (\Phi_1(\alpha), \Phi_2(\alpha))$ and $\Phi_k(\alpha) = (\phi_{1k}(\alpha), ..., \phi_{Ik}(\alpha), y_{1k}(\alpha), ..., y_{Ik}(\alpha))$, where N and $(\boldsymbol{\phi}(\alpha), \boldsymbol{y}(\alpha))$ are given at the end of the proof of Lemma 1. By the implicit function theorem, the Jacobian matrix $\partial\Phi_k$ of Φ_k evaluated at 1 is $\partial\Phi_k = -\partial F^{-1}_{\phi yk}\partial F_{\alpha k}$, where $\partial F^{-1}_{\phi yk}$ is the inverse of $\partial F_{\phi yk}$ in (34) and $\partial F_{\alpha k}$ is the Jacobian matrix of

$(F_{1k}, ..., F_{Ik})$ for F_{ik} in (32) and (33) with respect to α evaluated at $(\boldsymbol{\phi}, \boldsymbol{y}, \alpha) = (\boldsymbol{\phi}^\circ, \boldsymbol{y}^\circ, 1)$. Because

$$\partial F_{\alpha k} = \begin{bmatrix} -y^\circ u'(y^\circ)\mathbf{I} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & y^\circ c'(y^\circ)\mathbf{I} \end{bmatrix},$$

$\partial\Phi_k$ is invertible and so is the Jacobian matrix $\partial\Phi$ of Φ evaluated at 1. Next define the mapping $(\boldsymbol{\phi}, \boldsymbol{y}) \mapsto \Omega(\boldsymbol{\phi}, \boldsymbol{y})$ from $S' = \{(\boldsymbol{\phi}, \boldsymbol{y}) \in \mathbb{R}^{4I}_{++} : \phi_{i1}\phi_{12} = \phi_{11}\phi_{i2}\}$ to $\mathbb{R}^{I-1}$ by $\Omega_i(\boldsymbol{\phi}, \boldsymbol{y}) = \phi_{i1}\phi_{12} - \phi_{11}\phi_{i2}$ for $2 \leq i \leq I$. Apparently, the Jacobian matrix $\partial\Omega$ of $\Omega(\boldsymbol{\phi}, \boldsymbol{y})$ has full rank $I-1$ ($\partial\Omega_i/\partial\phi_{i1} = \phi_{12}$, $\partial\Omega_i/\partial\phi_{i2} = -\phi_{11}$, $\partial\Omega_i/\partial\phi_{jk}$ vanishes if $j \neq i$, and $\partial\Omega_i/\partial y_{jk}$ vanishes all (i,k)). Finally, note that $S = \{\alpha : \phi_{i1}(\alpha)\phi_{12}(\alpha) = \phi_{11}(\alpha)\phi_{i2}(\alpha)\}$ is the zero set of the composition mapping $\Omega \cdot \Phi$ from N to $\mathbb{R}^{I-1}$. Because the product of $\partial\Omega$ and $\partial\Phi$ has the full rank $I-1$, $0 \in \mathbb{R}^{I-1}$ is a regular value of $\Omega \cdot \Phi$. Then by the preimage theorem, dim $S = 3I + 1$.

Non-genericity of $\varphi_{i1}E_h\varphi_{j2} = \varphi_{i2}E_h\varphi_{j1}$ in Proposition 2

Fix (i, h). Let $\Gamma(\alpha) = y_{i1}c'_{i1}(y_{i1})E_h y_{j2}c'_{j2}(y_{j2}) - y_{i2}c'_{i2}(y_{i2})E_h y_{j1}c'_{j1}(y_{j1})$, where y_{jk} is an implicit function of α determined by $c'_{jk}(y_{jk}) = u'_{jk}(y_{jk})$. If $h \neq i$ then $\partial\Gamma/\partial\rho_{h1} = -\varphi_{i2}\pi_{hh}[c'_{h1}(y_{h1}) + y_{h1}c''_{h1}(y_{h1})]\partial y_{h1}/\partial\rho_{h1}$. If $h = i$ then for $h' \neq h$, $\partial\Gamma/\partial\rho_{h'1} = -\varphi_{i2}\pi_{hh'}[c'_{h'1}(y_{h'1}) + y_{h'1}c''_{h'1}(y_{h'1})]\partial y_{h'1}/\partial\rho_{h'1}$. Because $\partial y_{j1}/\partial\rho_{j1} = -c'(y_{j1})[c''_{j1}(y_{j1}) - u''_{j1}(y_{j1})]^{-1}$, the Jacobian of Γ evaluated at any α has full rank. So the dimension of the zero set of Γ is $4I - 1$.

Non-genericity of $\beta_{flex} = \beta_{fix}$ in Proposition 3

Fix i. Let $\Gamma_k(\alpha) = \sum_{t\geq 1}\sum_j \beta^t\pi_{ij}(t)[u_{jk}(y_{jk}) - c_{jk}(y_{jk})] - c_{ik}(y_{ik})$, where y_{jk} is an implicit function of α determined by $c'_{jk}(y_{jk}) = u'_{jk}(y_{jk})$. Let $\Gamma(\alpha) = \Gamma_1(\alpha) - \Gamma_2(\alpha)$. Fix $j \neq i$. Using $\partial\Gamma/\partial\theta_{j1} = \sum_{t\geq 1}\sum_j \beta^t\pi_{ij}(t)[u(y_{j1}) + u'_{j1}(y_{j1}) - c'_{j1}(y_{j1})]\partial y_{j1}/\partial\theta_{j1}$, we have $\partial\Gamma/\partial\theta_{j1} = \sum_{t\geq 1}\sum_j \beta^t\pi_{ij}(t)u(y_{j1})\partial y_{j1}/\partial\theta_{j1}$. Because $\partial y_{j1}/\partial\theta_{j1} = u'(y_{j1})[c''_{j1}(y_{j1}) - u''_{j1}(y_{j1})]^{-1}$, the Jacobian of Γ evaluated at any α has full rank. So the dimension of the zero set of Γ is $4I - 1$.