On Nonneutrality of the Exchange-Rate Regime^{*}

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Abstract

The presumption that flexible internal prices render the exchange-rate regime neutral in the presence of the aggregate shocks overlooks the channel that the current nominal quantity of a currency may affect current output through its (expected and real) future value. Employing a two-country variant of the Lagos-Wright model, we show that this channel undermines the presumption. We further examine whether the world may benefit from integration of separated values of two currencies by fixing exchange rates in a setup with country-specific cash-in-advance (CIA) constraints and a setup that endogenously eliminates the Kareken-Wallace indeterminacy absent country-specific CIA constraints.

Key words: Flexible prices; exchange-rate regimes; unified currency; optimal currency areas; neutrality

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1 Introduction

There is a presumption that the flexible exchange rate regime (the flexible-rate regime hereafter) and the fixed exchange rate regime (the fixed-rate regime hereafter) are equivalent if prices are flexible:

If internal prices were as flexible as exchange rates, it would make little economic difference whether adjustments were brought about by changes in exchange rates or equivalent changes in internal prices (Friedman [5, page 165]).)

This view, labeled exchange-rate-regime neutrality or simply *neutrality*, is the foundation of the literature on optimum currency areas (OCA) and most of the subsequent literature that compares a unified currency to a system of country-specific currencies under flexible exchange rates (e.g., Krugman [19] and Mundell [26]). A formal statement of neutrality is in Lucas [23] by an endowment-economy model with countryspecific cash-in-advance (CIA) constraints and a complete asset market. Although it is known that neutrality can fail with an incomplete asset market (see Helpman and Razin [10] and Lahiri et al [21]), neutrality remains a favored proposition in the literature (e.g., Mussa [28] and Obstfeld and Rogoff [30]) and appears to have a persuasive rationale. Think of that the US and EU are subject to some aggregate shocks which shift the demands for the dollar and euro; the current shock causes a stronger-than-average demand for the dollar. Under the flexible-rate regime, the dollar appreciates above its average while the quantity of each currency is kept at a fixed average. Under the fixed-rate regime, the quantity of dollars rises above its average while the quantity of euros falls. The fixed-rate regime generates more dollars and less euros, but each economy ends up with the same real balances under the two regimes because of flexible internal prices, leading to neutrality.

The asset market being complete or not, this rationale for neutrality seems incoherent in the presence of a *production channel*—people earn money for their future consumption by supplying labor elastically. Why? Suppose neutrality holds. With the production channel, how much one dollar or euro can incentivize people to work *now* ought to depend on its expected and real *future* value. For neutrality to hold in future, this future value of each currency ought to be the same under both regimes. But, then, neutrality falls apart—more dollars generated by the fixed-rate regime ought to induce higher current labor input and more current output in the US after being translated into higher nominal labor earnings, and less euros ought to induce less current output in the EU. The production channel apparently does not present in an exchange-rate model with no production (e.g., the Lucas [23] model and the money-in-the-utility-function (MIUF) model of Obstfeld and Rogoff [30, section 8.7]), with no currencies (e.g., Gabaix and Maggiori [6] and Itskhoki and Muzhin [13]), or with just one period. It calls us to examine whether the worldwide production can benefit from integration of separated values of multiple currencies into a unified value by fixing exchange rates or, equivalently, by a unified currency; this is a new perspective for comparing the fixed-rate regime to the flexible-rate regime and a unified currency to multiple currencies.

To facilitate deduction of equilibrium properties, we place the production channel in a two-country variant of the Lagos-Wright (LW) [20] model similar to the one in Gomis-Porqueras et al [7].¹ In this model, each country produces a single good traded in its domestic market. Prior to trading goods for currencies in domestic markets, people trade currencies and a linear good in a worldwide market to respond to an aggregate shock with country-specific effects; the trade of the linear good plays the role of the asset market in the Lucas [23] model. We consider two setups of domestic markets. The first, which is close to the Lucas [23] model, has competitive markets with country-specific CIA constraints. The second has people trading in pairs; our equilibrium concept is that the outcome of a pairwise meeting is in the pairwise core (see Wallace [33]),² which endogenously eliminates the Kareken-Wallace [14] indeterminacy (see Rocheteau and Nosal [31]). We use the first setup to illustrate that the exchange-rate regime is nonneutral by showing that the two different regimes support different sets of equilibrium allocations.

We use both setups to analyze which regime is better, i.e., admits a larger parameter space, to sustain the worldwide efficient production. In the first setup, efficiency is obtained when the rates of return of currencies are stabilized under the Friedman rule financed by lump sum taxes. When the tax enforcement is through excluding

¹These authors focus on quantitative analysis of flexible exchange rates. An alternative model would be the large-household model used by Head and Shi [9].

²For the first setup, as is well known, the domestic-market outcome is not in the core for all market participants. If we interpret the first setup as one that people trade in pairs subject to competitive pricing (see Rocheteau and Wright [32]) or if we impose country-specific CIA constraints in the second setup and let the pairwise trade be determined by generalized Nash bargaining, then the meeting outcome is not in the pairwise core.

a person from the economy, policy is constrained because the stabilization cost, i.e., the cost of paying current taxes to stabilize rates of return of currencies, cannot exceed the continuation payoff for the person to stay. Conditional on that all other people always stay, the two regimes share the same continuation payoff at any state to incentivize an individual person to stay. Provided that the unified current value of two currencies obtained by fixing exchange rates is stable across relevant states, the fixed-rate regime is better—the variation in the separated value of each currency increases the variation in the stabilization cost for the flexible-rate regime. But no regime is unambiguously better than another.

In the second setup, efficiency is attained for patient people absent lump sum taxes. Here fixing exchange rates has two opposite effects: a risk-sharing effect and a value-losing effect. For risk sharing, think of that the separated future value of the dollar is insufficient to support the current efficient output in the US but the unified future value of the euro and dollar is sufficient to support the current worldwide efficient output. Value losing means that the unified future value of the euro and dollar is lower than the sum of their separated future values, and it may be regarded as a consequence of inflexibility of the fixed-rate regime. Indeed, when the aggregate shock makes the current pairwise consumption-production condition in the US more attractive than in the EU, the dollar should have been more valuable than the euro on the worldwide market, but fixing exchange rates renders the EU's condition as the one that determines the current values of two currencies; in general equilibrium, this effect on the unified current value of the euro and dollar passes to their unified future value. We find that the fixed-rate regime is better for a class of familiar preferences, but, again, no regime is unambiguously better. To our best knowledge, this is the first result in the literature showing that one regime can dominate another in the absence of country-specific CIA constraints.

Related literature

Helpman and Razin [10] study a small-open-endowment-economy model with flexible prices, bonds, and aggregate shocks. In their model, when the aggregate shock is realized, a person (along the equilibrium path) in the small economy need not adjust his portfolio under the flexible-rate regime but desires to do so under the fixed-rate regime; but because the asset market is incomplete (i.e., the bonds market is close at this time), the exchange-rate regime affects the constraint on the individual choice problem and neutrality fails. In a related model, Lahiri et al [21] show that limited participation to the asset market can cause failure of neutrality as the incomplete asset market. Our point is that independently of how the asset market is arranged, the neutrality proposition is incoherent in the presence of the production channel.

In the foundation for the OCA literature, Mundell [26] sees lower transaction costs as the benefit of a unified currency and the inability to run country-specific stabilization policy as the cost when internal prices are sticky. Relating to Mundell [26] from a modern monetary aspect, Cooper and Kempef [3] formulate an OLG model with flexible prices which captures the transaction cost of multiple currencies by an infinite cost for old people to adjust portfolios of currencies and the policy inflexibility of a unified currency by the equal sharing rule of inflation taxes between two countries. Our model does not have any exogenous transaction cost or policy inflexibility, while our first setup admits an endogenous cost to running the Friedman rule. Although the Friedman rule as a stabilization policy may be special, our study adds to the cost-benefit analysis of the OCA literature a seemingly general factor—policy, being country-specific or not, has an endogenous cost to operate.

According to McKinnon [25], the monetary framework developed by Mundell [27] represents the genuine argument of Mundell for the euro. The essential point of Mundell [27] is that a unified currency better buffers the aggregate shocks by keeping the purchasing power of all nominal assets stable. Motivated by Mundell [27], Ching and Devereux [2] study a one-period MIUF model in which two currencies prohibit the international risk sharing because only home currency enters into one's utility function and a unified currency, which is always home currency, permits. In the second setup of our model, fixing exchange rates of two currencies promotes risk sharing but incurs a value-losing effect, an effect that has no counterpart in Ching and Devereux [2] and adds a novel piece to the framework of Mundell [27].

A group of papers concern whether one or two currencies better facilitate bilateral trades. Matsuyama et al [24] make the point that when there is no intrinsic difference between currencies, restrictions on which currency may be used in certain trades do not help. Kocherlakota and Krueger [18] exploit the signaling advantage of two currencies in the presence of asymmetric information. Kocherlakota [17] and Dong and Jiang [4] emphasize the gain of two currencies in expanding the recording-keeping dimension. Kiyotaki and Moore (2002) find that one currency may hurt specialization. Araujo and Ferrais [1] show that two currencies help shift misallocated liquidity. All these papers abstract away aggregate shocks.

2 A two-country variant of the LW model

There are two countries, 1 and 2. Each is populated by a nonatomic unit measure of infinitely-lived people and has its own currency. Each date has two stages; each stage has a produced and perishable good. Actions on a date are depicted here:

Individual		Three		Stage 1		Stage 2
person	\rightarrow	shocks	\rightarrow	linear good	\rightarrow	trade in each
$m = (m_1, m_2)$		realized		and two currencies		$\operatorname{country}$

Specifically, each person enters a date with a portfolio $m = (m_1, m_2) \in \mathbb{R}^2_+$, where m_k is the amount of currency k, i.e., country k's currency, for $k \in \{1, 2\}$. Then, three shocks, two idiosyncratic shocks and one aggregate shock, are realized. One idiosyncratic shock determines a person to be a *producer* or a *consumer* at stage 2 for one date with equal probability; another determines a constant fraction λ of consumers in each country to be *tourists* at stage 2 for one date. The aggregate shock determines the current aggregate state; there are I aggregate states. The transition of states follows a Markov chain with a positive *transition matrix* $\pi = (\pi_{ij})$.

At stage 1, everyone can produce and consume a linear good, i.e., one's utility from consuming q is q and from producing q is -q. At stage 2, a producer produces and a nontourist consumer consumes in home; a tourist consumes in foreign country and returns to home at the end of the date; and real international trade consists solely of tourism. When the current (aggregate) state is $i \in \{1, 2, ..., I\}$, the utility of a consumer who consumes $q \ge 0$ in country k is $u_{ik}(q) \equiv \theta_{ik}u(q)$ and the disutility of a producer who produces q in country k is $c_{ik}(q) \equiv \rho_{ik}c(q)$, where $\theta_{ik}, \rho_{ik} > 0, u(0) =$ $c(0) = 0, u' > 0, u'' < 0, c' > 0, c'' \ge 0, \beta u'(0) > c'(0)$, and β is the discount factor. Each person's period utility is the sum of his stage-1 utility and stage-2 utility; he maximizes expected discount utility. We refer to $\alpha = (\theta_{i1}, \rho_{i1}, \theta_{i2}, \rho_{i2})_{i=1}^{I}$ as the *shock vector* and represent the aggregate shock by (π, α) . Say a shock (π, α) is symmetric if each i has a symmetric state $\sigma(i)$, i.e., there is a mapping $\sigma : \{1, 2, ..., I\} \rightarrow \{1, 2, ..., I\}$ such that $\sigma(\sigma(i)) = i$ and $(\alpha_{\sigma(i)1}, \alpha_{\sigma(i)2}) = (\alpha_{i2}, \alpha_{i1})$ all $i, \pi_{ij} = \pi_{\sigma(i)\sigma(j)}$ all (i, j), and $\sigma(i) = i$ at most one i.

Stage 1 has a worldwide competitive market for everyone to trade the two currencies and the linear good, where the exchange rate, i.e., the relative price of two currencies, is either flexible or fixed. When the exchange rate is fixed, the government of each country is committed to supply unlimited amounts of its own currency, and we normalize the fixed value of the exchange rate as unity. The governments run statecontingent policy $\gamma = (\gamma_{i1}, \gamma_{i2})_{i=1}^{I}$, which withdraws $(1 - \gamma_{ik})s_{ik}$ amount of currency k in the coming stage-1 market when i is the current state and s_{ik} is the amount of currency k at the end of the current stage 1. A policy is *inactive* if $\gamma_{ik} = 1$ all (i, k)and *active* otherwise. When the policy is active, each person has a country-specific identity for governments to collect country-specific lump sum taxes. We renormalize the nominal quantities after policy is implemented so that $s_{i1} = s_{i2} = 0.5$ under the flexible rate regime and $s_{i1} + s_{i2} = 1$ under the fixed rate regime at the end of each stage 1. Stage 2 has a domestic market in each country to trade the stage good.

We limit consideration to stationary equilibria in that when i is the current state, the amount of goods ϕ_{ik} spent on the current stage-1 market to acquire the amount of currency k that is equal to one (renormalized) unit at the end of stage 1 depends only on i. We refer to $\phi = (\phi_{i1}, \phi_{i2})_{i=1}^{I}$ as a stage-1 price vector. Under the flexible-rate regime and the fixed-rate regime, respectively, the per unit prices of currency k in the coming stage-1 market are $(\gamma_{ik})^{-1}\phi_{jk}$ and $(s_{i1}\gamma_{i1} + s_{i2}\gamma_{i2})^{-1}\phi_{jk}$ when j is the next state, and, therefore, the (gross expected) rates of return from carrying currency kinto the coming stage-1 market are

$$\zeta_{ik} = (\gamma_{ik})^{-1} E_i(\phi_{jk}/\phi_{ik}) \text{ and } \zeta_{ik} = (s_{i1}\gamma_{i1} + s_{i2}\gamma_{i2})^{-1} E_i(\phi_{jk}/\phi_{ik}), \tag{1}$$

where E_i stands for the expectation made at state *i*. As is well known, linearity of the stage-1 good implies that the continuation payoff $w_{ik}(m)$ for a country-*k* resident who leaves the current stage 2 with the portfolio *m* takes the form

$$w_{ik}(m) = \beta(m_1\phi_{i1}\zeta_{i1} + m_2\phi_{i2}\zeta_{i2}) + A_{ik}$$
(2)

for some constant A_{ik} ; moreover, by (2), ζ_{ik} cannot exceed $1/\beta$ in any equilibrium.

A stage-2 allocation $y = (y_{i1}, y_{i2})_{i=1}^{I}$ or simply an allocation is a positive vector consisting of stage-2 output y_{ik} of each country k for each state i. We focus on stage-2 output because it determines ex ante welfare of people. Let y^* denote the efficient allocation such that $u'_{ik}(y^*_{ik}) = c'_{ik}(y^*_{ik})$ all (i, k); we focus on the values of α with $y^*_{ik} >$ 0 all (i, k). An allocation is supported by a regime if it is an equilibrium allocation under the regime; the exchange-rate regime is neutral if the two regimes support the same set of allocations and non-neutral otherwise. The complete description of the equilibrium conditions depends on the form of the stage-2 domestic markets. We consider two forms in the next two sections.

3 Competitive stage-2 markets

In this section, each stage-2 domestic market is competitive and subject to countryspecific CIA constraint—currency k can only be used in country k. Because ζ_{ik} , the rate of return of currency k, cannot exceed $1/\beta$, it is without loss of generality to assume that only consumers who will consume at country k enter stage 2 with currency k. If such a consumer resides in country l and enters stage 2 with m_k units of currency k in state i, then given v_{ik} as the per unit price of currency k in the stage-2 country-k market, his continuation payoff is

$$W_{ik}(m_k; l) = \max_{(q, m'_k)} u_{ik}(q) + \beta m'_k \phi_{ik} \zeta_{ik} + A_{il}, \ s.t. \ 0 \le q + v_{ik} m'_k \le v_{ik} m_k;$$
(3)

thus, his trade in the stage-1 market must lead him to enter stage 2 with

$$m_{ik} = \arg \max_{m_k \ge 0} [-m_k \phi_{ik} + W_{ik}(m_k; l)].$$
(4)

To clear the stage-1 market, we need $m_{i1} = m_{i2} = 1$ for the flexible-rate regime and $m_{i1}+m_{i2} = 2$ for the fixed-rate regime. Let y_{ik} denote the optimal q for the consumer's problem in (3) with $m_k = m_{ik}$. By the envelope condition, $\phi_{ik} = v_{ik}u'_{ik}(y_{ik})$. To clear the stage-2 market, a producer in country k must produce y_{ik} so $v_{ik}c'_{ik}(y_{ik}) = \beta \phi_{ik}\zeta_{ik}$. When $\beta \zeta_{ik} < 1$, the CIA constraint $y_{ik} \leq v_{ik}m_{ik}$ in (3) must bind; when $\beta \zeta_{ik} = 1$, it is without loss of generality to limit attention to equilibria with $y_{ik} = v_{ik}m_{ik}$. Hence the consumer's optimal conditions in the country-k stage-2 market, respectively, can be written as

$$m_{ik}\phi_{ik} = y_{ik}u'_{ik}(y_{ik}),\tag{5}$$

$$y_{ik}c'_{ik}(y_{ik}) = \beta m_{ik}\phi_{ik}\zeta_{ik}.$$
(6)

Definition 1 Given policy γ , a flexible-rate (fixed-rate, resp.) equilibrium is a positive stage-1 price vector ϕ such that (5) and (6) hold for some allocation y when $m_{ik} = 1$ all (i, k) $(m_{i1} + m_{i2} = 2$ and $\phi_{i1} = \phi_{i2}$ all i, resp.).

3.1 Example

Let I = 2, the aggregate shock be symmetric, and the policy be inactive. Symmetry in the shock implies symmetry in equilibrium outcomes, i.e., $(a_{21}, a_{22}) = (a_{12}, a_{11})$ for $a = \phi, m$, and y. Let $c(q) = 0.5q^2$ and $u(q) = \ln q + L > 0$ if $q \ge q > 0$, where L and \underline{q} are constant and \underline{q} is small.³ Now the consumer's optimal condition (5) is $m_{ik}\phi_{ik} = \theta_{ik}$ and the producer's optimal condition (6) is $\rho_{ik}y_{ik}^2 = \beta m_{ik}E_i\phi_{jk}$. Set $\theta_{11} + \theta_{12} = 2$, $\rho_{11} + \rho_{12} = 2$, $|\theta_{11} - 1| \in (0, 1 - \beta)$, and $\pi_{11} = 0.5$. In the flexible-rate equilibrium, $m_{11} = m_{12} = 1$. So by (5), we have $(\phi_{11}, \phi_{12}) = (\theta_{11}, \theta_{12})$ and $(E_1\phi_{j1}, E_1\phi_{j2}) = (1, 1)$; then by (6),

$$(y_{11}, y_{12}) = (\sqrt{\beta/\rho_{11}}, \sqrt{\beta/\rho_{12}}).$$
 (7)

In the fixed-rate equilibrium, $m_{11}+m_{12}=2$ and $\phi_{11}=\phi_{12}$. So by (5) and $\theta_{11}+\theta_{12}=2$, we have $\phi_{11}=\phi_{12}=1$, $(m_{11},m_{12})=(\theta_{11},\theta_{12})$, and $(E_1\phi_{j1},E_1\phi_{j2})=(1,1)$; by (6),

$$(y_{11}, y_{12}) = (\sqrt{\beta \theta_{11}/\rho_{11}}, \sqrt{\beta \theta_{12}/\rho_{12}}).$$
 (8)

Here the shock shifts demands for currencies solely by its influence on the coefficient θ of the utility function u (a feature of u used in the example); the flexible-rate regime responds with changes in prices $((\phi_{11}, \phi_{12}) = (\theta_{11}, \theta_{12}))$ while the fixed-rate regime with changes in quantities $((m_{11}, m_{12}) = (\theta_{11}, \theta_{12}))$. The future value of one unit of each currency is equal to unity under both regimes so the difference of outputs between (7) and (8) is completely determined by different quantities of currencies generated by the two regimes.

As the exchange-rate regime affects output, it ought to affect the real exchange rate. In this example, as detailed in the appendix, the flexible-rate regime can experience greater volatility of the real exchange rate than the fixed-rate regime. This is qualitatively consistent with the renowned finding of Mussa [28], often viewed as evidence for monetary nonneutrality. Our exercise suggests that Mussa's finding need not invalidate neutrality of money but may support exchange-rate-regime nonneutrality.

What if the shock does not shift the demands for currencies (even with its influence on θ)? For this to happen, let the two currencies and the linear good be traded in the stage-1 market *before* any shock is realized, and let the two currencies be traded in a *foreign-exchange market* after all shocks are realized but before stage 2 starts; these two markets have the same exchange-rate regime. Now in equilibrium, the price of currency k in the stage-1 market is a constant q_k and each person leaves the market with the portfolio (0.5, 0.5). For the flexible-rate regime, let us focus on equilibria

³For $0 \le q < \underline{q}$, we can set $u(q) = 2(y/\underline{y})^{0.5} + \ln \underline{q} + L - 2$, which satisfies u(0) = 0, a property used in Propositions 2 and 3 below. All derivations in the example go through if $u(q) = \ln q$ all q > 0.

that there is no arbitrage gain for producers between the foreign-exchange market and the coming stage-1 market. Then under each regime, a person can by trading on the foreign-exchange market carry the amount of currency k worth of $q = 0.5(q_1 + q_2)$ units of goods in the coming stage-1 market. So by a condition analogous to (5), $q = 0.25(1 - \lambda)(\theta_{11} + \theta_{21}) + 0.25\lambda(\theta_{12} + \theta_{22}) + 0.5\beta q$. Then $q = 0.5/(1 - 0.5\beta)$ and neutrality follows from $\rho_{ik}y_{ik}^2 = \beta q$, a condition analogous to (6).⁴

3.2 Results

We present two results here. The first result is that the exchange-rate regime is nonneutral in a generic sense. Given the transition matrix π , genericity refers to a neighborhood of $1 \in \mathbb{R}^{4I}$ that contains the shock vector α . We first show that there exists a unique flexible-rate equilibrium $\phi(\alpha)$ for the inactive policy when the neighborhood is small (smallness ensures $\zeta_{ik} \leq 1/\beta$) and next show that when α is generic, no fixed-rate equilibrium (given any policy) supports the same allocation as the flexible-rate equilibrium $\phi(\alpha)$.

Lemma 1 Given π and some mild regularity condition, there exists a unique flexiblerate equilibrium $\phi(\alpha)$ for the inactive policy if α is in a neighborhood of $1 \in \mathbb{R}^{4I}$.

The proof of Lemma 1 is in the appendix. The proof is standard. The optimal conditions (5) and (6) with $m_{ik} = 1$ constitute 4*I* nonlinear equations in unknown (ϕ, y) . As is well known, when $\alpha = 1$, the 4*I* equations have a unique solution $(\phi^{\circ}, y^{\circ})$, where $y_{ik}^{\circ} = q^{\circ}$ and $\phi_{ik}^{\circ} = q^{\circ}u'(q^{\circ})$ all (i, k) and $\beta u'(q^{\circ}) = c'(q^{\circ})$. Then by the implicit function theorem, there exists a unique solution $(\phi(\alpha), y(\alpha))$ for α around 1.

Proposition 1 Given π , the set of α in a neighborhood of $1 \in \mathbb{R}^{4I}$ that permits the two regimes to support the same set of allocations is a measure-zero set.

Proof. For neutrality to hold, some fixed-rate equilibrium must support the same allocation y as the Lemma-1 flexible-rate equilibrium $\phi(\alpha)$. By (5) and (6), $\beta \zeta_{ik} u'_{ik}(y_{ik}) = c'_{ik}(y_{ik})$ in each of these two equilibria; that is, stage-2 country-k output in an equilibrium is pinned down by the rate of return of currency k at each state.

⁴The flexible exchange rate, however, is indeterminate because producers can meet the extra supply or demand of any currency on the part of consumers in the foreign-exchange market when $q_1 + q_2 = 1 + \beta$ and q_1 and q_2 are close to each other. This resembles the finding of King et al [16] when there are no (intrinsic) aggregate shocks.

Because the rates of return of two currencies are equal at each state in the fixed-rate equilibrium, they must be equal in the equilibrium $\phi(\alpha)$. In $\phi(\alpha)$, $\zeta_{ik} = E_i \phi_{jk} / \phi_{ik}$; without loss of generality, let $\phi_{12}/\phi_{11} \ge \phi_{j2}/\phi_{j1}$ all j so

$$\zeta_{11} - \zeta_{12} = \sum_{j} \pi_{1j} \left(\frac{\phi_{j1}}{\phi_{11}} - \frac{\phi_{j2}}{\phi_{12}} \right) = \sum_{j} \frac{\pi_{1j} \phi_{j1}}{\phi_{12}} \left(\frac{\phi_{12}}{\phi_{11}} - \frac{\phi_{j2}}{\phi_{j1}} \right) = 0 \tag{9}$$

holds only if the exchange rate ϕ_{j2}/ϕ_{j1} is constant in j.⁵ As verified in the appendix, the Jacobian matrix of the mapping $\alpha \mapsto (\phi(\alpha), y(\alpha))$ evaluated at $\alpha = 1$ has full rank. Then by the pre-image theorem, the set $S = \{\alpha : \phi_{i1}(\alpha)\phi_{12}(\alpha) = \phi_{11}(\alpha)\phi_{i2}(\alpha)\},$ viewed as a manifold, is 3I + 1.

The intuition behind the proof of Proposition 1 is simple. Neutrality forces the flexible-rate equilibrium $\phi(\alpha)$ to fix the exchange rate for (9) to hold. But because the equilibrium outcome $(\phi(\alpha), y(\alpha))$ should vary with the physical environment α , fixing the exchange rate limits the freedom of $\phi(\alpha)$ to vary and, hence it can only happen for a measure-zero set of α .⁶ Remarkably, the proposition does not hold if producers supply goods inelastically (e.g., they sell endowed goods as workers in the shopper-worker pairs in the Lucas [23] model); in that case, (6) is not an equilibrium condition so the flexible-rate equilibrium is not required to have a fixed exchange rate when it supports the same allocation as a fixed-rate equilibrium.

Now we turn to our second result, which pertains to the optimal regime. We begin with the observation that the flexible-rate regime can imitate the fixed-rate regime to support a same allocation by adopting a suitable active policy. Let ϕ be a fixed-rate equilibrium given policy γ . Let $m'_{ik} = 1$ and let the imitating flexible-rate equilibrium ϕ' equate the *current* values of m_{ik} and m'_{ik} in the two regimes, i.e.,

$$m_{ik}'\phi_{ik}' = m_{ik}\phi_{ik};\tag{10}$$

let the imitating policy γ' equate the *future* values of m_{ik} and m'_{ik} , i.e., $\zeta'_{ik}m'_{ik}\phi'_{ik} = \zeta_{ik}m_{ik}\phi_{ik}$ or given (10), $\zeta'_{ik} = \zeta_{ik}$, which by (1) means

$$\gamma'_{ik} = (\phi'_{ik}\zeta_{ik})^{-1} E_i \phi'_{jk}.$$
(11)

⁵This line of argument is pointed out to us by Harald Uhlig.

⁶The measure-zero set contains any α with $(\theta_{i1}, \rho_{i1}) = (\theta_{i2}, \rho_{i2})$ all *i* but it may contain other α (e.g., α with $\theta_{11} = \theta_{12}$ in the example of section 3.1). On a separated note, the proof of the proposition is adaptable if the number of state-dependent parameters in α increases, only θ_{ik} s are state-dependent, or only ρ_{ik} s are state-dependent and $q \mapsto qu'(q)$ is strict monotonic around q° .

In fact, (10) and (11) are necessary and sufficient for (5) and (6) to hold when $(m'_{ik}, \phi'_{ik}, \zeta'_{ik})$ is substituted for $(m_{ik}, \phi_{ik}, \zeta_{ik})$. Thus flexibility of choosing currency-specific policy appears to at least allow the flexible-rate regime to be not dominated by the fixed-rate regime. But is it really so?

When taxes are positive in equilibrium, there must be sufficient coercive power for enforcement. Such power is assumed in Definition 1. But consider the scenario that after a person refuses to pay the current taxes, the most severe punishment is to exclude him from all future market activities after the current stage-1 market is closed. Then the coercive power is apparently endogenous and there should be a taxation constraint on equilibrium.

To describe the taxation constraint, fix a policy-equilibrium pair (γ, ϕ) and let *i* be the *current* state and *h* be the *previous* state. If a country-*k* resident pays the current taxes τ_{hik} , then his continuation payoff is $-c_{ik}(y_{ik}) + \beta m_{ik}\phi_{ik}\zeta_{ik} + A_{ik}$ as a producer and $u_{il}(y_{il}) - m_{il}\phi_{il} + A_{ik}$ as a consumer to consume in country *l*; using (5) and (6), $\tau_{hik} \leq U_{ik} + A_{ik}$ is necessary for all country-*k* residents to pay taxes, where $U_{ik} = \min\{-c_{ik}(y_{ik}) + y_{ik}c'_{ik}(y_{ik}), \min_{l \in \{1,2\}}\{u_{il}(y_{il}) - y_{il}u'_{il}(y_{il})\}\}$. Hence the total tax revenues $T_{hi} \equiv \sum_{k} \tau_{hik}$ for the pair (γ, ϕ) are constrained by

$$T_{hi} \le \sum_{k} U_{ik} + \sum_{k} A_{ik}.$$
(12)

Lemma 2 Let y be the allocation supported by the fixed-rate pair (γ, ϕ) and the flexible-rate pair (γ', ϕ') in (10) and (11). Let $\varphi_{jk} = y_{jk}u'_{jk}(y_{jk})$ and $\varphi_j = \sum_k \varphi_{jk}$. Let ζ_h denote the common value of ζ_{h1} and ζ_{h2} in the fixed-rate equilibrium ϕ . Then (γ, ϕ) and (γ', ϕ') , respectively, demand the tax revenues

$$T_{hi} = 0.5[\zeta_h \varphi_i \varphi_h (E_h \varphi_j)^{-1} - \varphi_i] \text{ and } T'_{hi} = 0.5[\zeta_h \sum_k \varphi_{ik} \varphi_{hk} (E_h \varphi_{jk})^{-1} - \varphi_i]$$
(13)

to withdraw currencies; moreover, when y is the efficient allocation y^* , the taxation constraints (12) for (γ, ϕ) and (γ', ϕ') , respectively, can be written as

$$g_{hi} \equiv \varphi_i \varphi_h (E_h \varphi_j)^{-1} \le V_i(\beta) \text{ and } f_{hi} \equiv \sum_k \varphi_{ik} \varphi_{hk} (E_h \varphi_{jk})^{-1} \le V_i(\beta), \qquad (14)$$

where $V_i(\beta) = 2\beta \sum_k U_{ik} + \beta \sum_{t \ge 1} \sum_j \beta^t \pi_{ij}(t) \sum_k [u_{jk}(y_{jk}) - c_{jk}(y_{jk})]$ and $\pi_{ij}(t)$ is the *t*-step transition probability from state *i* to state *j*.

The proof of Lemma 2 is in the appendix. In Lemma 2, $\varphi_{jk} = m_{jk}\phi_{jk} = m'_{jk}\phi'_{jk}$ is the current value of all currency k at state j measured in the stage-1 goods unit for both regimes. Using $E_h a_i = E_h a_j$, we see $E_h T'_{hi} = E_h T_{hi} = 0.5(\zeta_h \varphi_h - E_h \varphi_i)$, i.e., the two regimes demand the same expected tax revenues before the current state *i* is revealed. After *i* is realized, the two regimes rely on the same continuation payoff for all people to stay and for each regime, this payoff must cover the different current tax costs stemming from different previous states. When $y = y^*$, policies γ and γ' are the Friedman rule which stabilizes the rate of return of each currency at any state at $1/\beta$, and $V_i(\beta)$ is the continuation payoff shared by all people at state *i*.

Examining Lemma 2, we see that $V_i(0) = 0$, $V_i(1) = \infty$ and $V'_i(\beta) > 0$. Therefore, we get two values of the discount factor, $\beta_{flex}(h,i)$ and $\beta_{fix}(h,i)$, well defined by $f_{hi} = V_i(\beta_{flex}(h,i))$ and $g_{hi} = V_i(\beta_{fix}(h,i))$. Let $\beta_{flex} = \max_{(h,i)} \beta_{flex}(h,i)$ and $\beta_{fix} = \max_{(h,i)} \beta_{fix}(h,i)$. Lemma 2 implies the following immediately.

Lemma 3 In the presence of the tax constraint, the efficient allocation y^* is supported by the flexible-rate regime iff $\beta \geq \beta_{flex}$ and by the fixed-rate regime iff $\beta \geq \beta_{fix}$.

When the two cutoff values β_{flex} and β_{fix} in Lemma 3 differ, one regime dominates another over a range of values of β (the fixed-rate regime dominates if $\beta_{flex} > \beta_{fix}$).

Proposition 2 Suppose (i) given π , α is outside a measure-zero set in \mathbb{R}^{4I} , and (ii.a) α implies a sufficiently small cross-state variation in $\sum_{k} y_{ik}^* u'_{ik}(y_{ik}^*)$ or (ii.b) (π, α) is sufficiently closed to a symmetric (π', α') with π' representing an i.i.d. process. Then $\beta_{flex} > \beta_{fix}$. In general, which regime dominates may depend on the aggregate shock and preferences.

Proof. To begin with, let α imply $\varphi_{i1}E_h\varphi_{j2} \neq \varphi_{i2}E_h\varphi_{j1}$ all (h, i), which, as verified in the appendix, is a generic property given π . Because

$$f_{hi} - g_{hi} = L_{hi} \left(\frac{\varphi_{h1}}{\varphi_{h2}} - \frac{E_h \varphi_{j1}}{E_h \varphi_{j2}}\right) \left(\frac{\varphi_{i1}}{\varphi_{i2}} - \frac{E_h \varphi_{j1}}{E_h \varphi_{j2}}\right)$$
(15)

for some $L_{hi} > 0$, $f_{hi} \neq g_{hi}$ all (h, i). To continue, suppose either (a) φ_i is constant in *i* or (b) the aggregate shock is symmetric and π represents an i.i.d. process. Then given any current state *i*, for any previous state *h*, there is a previous state *h'* satisfying $f_{h'i} > g_{hi}$, i.e., the flexible-rate regime incurs a higher tax spike than the fixed-rate regime at *i*. Indeed, when condition (a) holds, $g_{hi} = g_{ii}$; by (15), $f_{ii} > g_{ii}$ (h' = i). When condition (b) holds, $g_{\sigma(h)i} = g_{hi}$, ($\varphi_{h1}, \varphi_{h2}$) = ($\varphi_{\sigma(h)2}, \varphi_{\sigma(h)1}$), and the value of $E_h \varphi_{jk}$ does not depend on (h, k); then by (15), $\max\{f_{hi}, f_{\sigma(h)i}\} > g_{hi}$ (h' = h or $\sigma(h)$). Because *i* is arbitrary, $\beta_{flex} > \beta_{fix}$. Note that $\beta_{flex} > \beta_{fix}$ holds for $\tilde{\alpha}$ in a neighborhood of α satisfying condition (a) and for $(\tilde{\pi}, \tilde{\alpha})$ in a neighborhood of (π, α) satisfying condition (b). An example with $\beta_{flex} < \beta_{fix}$ is given in the appendix.

Conditions (a) and (b) in the proof of Proposition 2 are intuitive. Integration of the separated values φ_{1i} and φ_{2i} of two currencies into the unified value $\varphi_i = \varphi_{i1} + \varphi_{i2}$ by fixing exchange rates results in a constant cost at the current state to stabilizing the rates of return of currencies over all (previous) states or over each pair of symmetric states because φ_i is constant over all states (condition (a)) or over the pair (condition (b)). Because the variation in each separated value under the flexible-rate regime leads to the variation in the stabilization cost, fixing exchange rates is beneficial. We find no conditions that are easily described and verified (as conditions (a) and (b)) to ensure $\beta_{flex} < \beta_{fix}$. An obvious direction is that $f_{hi} < g_{hi}$ for some (h, i) that attains β_{flex} , which is the case in the example given in the appendix. Among others, the shock exerts much asymmetric effects on two countries in that example.

4 Decentralized stage-2 markets

In this section, each consumer randomly meets a producer in each stage-2 domestic market. Critical for this setup is which outcome is selected in a pairwise meeting. Following Wallace [33], we require that any selection be in the pairwise core. With this approach, the Kareken-Wallace [14] indeterminacy can be eliminated endogenously so we do not impose country-specific CIA constraints. Also, efficiency can be attained absent lump sum taxes for patient people so we only consider the inactive policy, which may be justified by the assumption that people are all anonymous.

To be formal, consider a consumer who holds a portfolio m of currencies and a producer who holds a portfolio m' when they meet in country k at state i. Let (q, κ, ι) denote a generic outcome for the meeting, where q is the producer's output, κ is the consumer's payment in the producer's home currency (i.e., currency k), and ι is the payment in the producer's foreign currency. Our equilibrium involves a stage-1 price vector $\phi = (\phi_{i1}, \phi_{i2})_{i=1}^{I}$ and a construct, referred to as a stage-2 trading rule and written as $\xi = (\xi_{i1}, \xi_{i2})_{i=1}^{I}$. The price vector ϕ and the implied continuation payoff function in (2) define the pairwise core for the consumer and producer, and the trading rule ξ selects an outcome $\xi_{ik}(m, m') \equiv (y_{ik}(m, m'), \kappa_{ik}(m, m'), \iota_{ik}(m, m'))$ from the pairwise core.

We concentrate on equilibria satisfying no currency substitution, which is represented by that at the end of stage 1 of each period, currency k is only carried by people who trade in country k at stage 2. It is convenient to denote by $\eta(b_k)$ a portfolio $b = (b_1, b_2)$ of currencies with $b_l = 0, l \neq k$. Let $\eta(m'_{ik})$ and $\eta(m_{ik})$ be the portfolio held by producers in country k and the portfolio held by consumers who will consume in country k after stage-1 trade, respectively; given the price vector ϕ and the trading rule ξ , $\eta(m'_{ik})$ and $\eta(m_{ik})$ must be the best response to each other, i.e.,

$$\eta(m_{ik}) \in \arg \max_{m=(m_1,m_2)} \{-m_k \phi_{ik} - m_l \phi_{il} + u_{ik} (y_{ik}(m,\eta(m'_{ik}))) + \beta [m_k - \kappa_{ik}(m,\eta(m'_{ik}))] E_i \phi_{jk} + \beta [m_l - \iota_{il}(m,\eta(m'_{ik}))] E_i \phi_{jl} \},$$

$$\eta(m'_{ik}) \in \arg \max_{m'=(m'_1,m'_2)} \{-m'_k \phi_{ik} - m'_l \phi_{il} - c_{ik} (y_{ik}(\eta(m_{ik}),m')) + \beta [m'_k + \kappa_{ik}(\eta(m_{ik}),m'))] E_i \phi_{jk} + \beta [m'_l + \iota_{il}(\eta(m_{ik}),m'))] E_i \phi_{jl}.$$
(16)

The stage-1 market clearing requires $m_{ik} + m'_{ik} = 1$ for the flexible-rate regime and $\sum_k (m_{ik} + m'_{ik}) = 2$ for the fixed rate regime.

Definition 2 A flexible-rate (fixed-rate, resp.) equilibrium is a pair of a positive stage-1 price vector ϕ and a stage-2 trading rule ξ such that (16) and (17) hold when $m_{ik} + m'_{ik} = 1$ all (i, k) ($\sum_{k} (m_{ik} + m'_{ik}) = 2$ and $\phi_{i1} = \phi_{i2}$ all i, resp.).

Following Hu et al [11], one can translate a Definition-2 equilibrium into an equilibrium of a planner's mechanism-design problem; the planner seeks an optimal equilibrium allocation that maximizes *ex ante* welfare of people. As Hu et al [11], we focus on the case that the optimal is y^* .

We begin with necessary conditions for an equilibrium (ϕ, ξ) to support y^* . Let $u_{ik}^* = u_{ik}(y_{ik}^*), c_{ik}^* = c_{ik}(y_{ik}^*), d_{ik} = u_{ik}^* - c_{ik}^*, c_i^* = 0.5 \sum_k c_{ik}^* d_i = 0.5 \sum_k d_{ik}$, and $d = (d_i)_{i=1}^I$. Consider a consumer with $\eta(m_{ik})$ and a producer with $\eta(m_{ik}')$ in a country-k meeting at state *i*. The producer leaves stage 1 with $\eta(m_{ik}')$ only if

$$m'_{ik}\phi_{ik} \le -c^*_{ik} + \beta [m'_{ik} + \kappa_{ik}(\eta(m_{ik}), \eta(m'_{ik}))] E_i \phi_{jk};$$
(18)

that is, the (discounted) future value of all money accumulated by him at stages 1 and 2 must cover his cost to producing y_{ik}^* plus stage-1 cost to acquiring m'_{ik} . Also, the consumer leaves stage 1 with $\eta(m_{ik})$ only if

$$m_{ik}\phi_{ik} \le u_{ik}^* + \beta [m_{ik} - \kappa_{ik}(\eta(m_{ik}), \eta(m'_{ik}))] E_i \phi_{jk};$$
 (19)

that is, his utility of consuming y_{ik}^* plus the future value of any unspend money must cover his stage-1 cost to acquiring m_{ik} . Summing over (18) and (19), we have

$$\phi_{ik} \le \frac{d_{ik}}{m_{ik} + m'_{ik}} + \beta E_i \phi_{jk}; \tag{20}$$

that is, the consumer-producer joint meeting surplus plus the future value of $m_{ik} + m'_{ik}$ must cover their joint stage-1 cost to acquiring $m_{ik} + m'_{ik}$. By (18) and $\beta E_i \phi_{jk} \leq \phi_{ik}$ (the rate of return of currency k cannot exceed $1/\beta$), $c_{ik} \leq \beta \kappa_{ik} (\eta(m_{ik}), \eta(m'_{ik})) E_i \phi_{jk}$. Because the payment $\kappa_{ik} (\eta(m_{ik}), \eta(m'_{ik}))$ is bounded above by m_{ik} , it follows that

$$c_{ik}^* \le \beta m_{ik} E_i \phi_{jk}; \tag{21}$$

that is, the future value of currency k carried by the consumer must cover the producer's stage-2 cost of production. The stage-1 market-clearing condition and (21) imply

$$c_{ik}^* \le \beta E_i \phi_{jk},\tag{22}$$

$$c_i^* \le \beta E_i \phi_{jk},\tag{23}$$

for the flexible-rate regime and the fixed-rate regime, respectively. To proceed, we first derive an upper bound on $\beta E_i \phi_{jk}$ from (20) for each regime and next from (22) or (23) a lower bound on β that is necessary for the respective regime to support y^* .

Lemma 4 Let $v_{ik}(\beta) = \sum_{t\geq 1} \sum_{j} \beta^t \pi_{ij}(t) d_{jk}$ ($\pi_{ij}(t)$ is the t-step transition probability from state i to j). Let $x_{ik}^* = d_{ik} + v_{ik}(\beta)$ and suppose y^* is supported by a flexible-rate equilibrium (ϕ, ξ). Then $\phi_{ik} \leq x_{ik}^*$ and $\beta E_i \phi_{jk} \leq v_{ik}(\beta)$.

Proof. Let a mapping $H : \{x = (x_{i1}, x_{i2})_{i=1}^{I} : x \ge \phi\} \to \mathbb{R}^{2I}$ be defined by $H_{ik}(x) = d_{ik} + \beta E_{ik} x_{jk}$. By (20) and $m_{ik} + m'_{ik} = 1$ (stage-1 market clearing), $\phi_{ik} \le d_{ik} + \beta E_i \phi_{jk}$ so $H(\phi) \ge \phi$. Because $H(x) \ge H(\phi)$ and H is a contraction mapping, H has a unique fixed point x° and $x^{\circ} \ge \phi$. By repeated substitution, $x_{ik}^{\circ} = d_{ik} + \beta E_i x_{jk}^{\circ}$ yields $\beta E_i x_{jk}^{\circ} = v_{ik}(\beta)$. By the definition of x_{ik}^* , we see that x^* is x° .

Lemma 5 There exist greatest vectors $\nu = (\nu_i)_{i=1}^I$ and $\delta = (\delta_i)_{i=1}^I$ that satisfy $\nu_i(\beta) = \sum_{t\geq 1} \sum_j \beta^t \pi_{ij}(t) \delta_j$ and $\delta_i = \min\{d_i, \max\{\nu_i(\beta), c_i^*\} \min_k(d_{ik}/c_{ik}^*)\}$, and $\nu_i(\beta)$ is strictly increasing and continuous in β . Let $z_i^* = \delta_i + \nu_i(\beta)$ and suppose y^* is supported by a fixed-rate equilibrium (ϕ, ξ) . Then $\phi_i \leq z_i^*$ and $\beta E_i \phi_j \leq \nu_i(\beta)$, where ϕ_j is the common value of ϕ_{j1} and ϕ_{j2} .

Proof. Existence of $\nu(\beta)$ and δ and properties of δ are shown in the appendix. Now let a mapping $G: \{z = (z_i)_{i=1}^I : z \ge \phi^\circ \equiv (\phi_i)_{i=1}^I\} \to \mathbb{R}^I$ be defined by

$$G_i(z) = \max_{(b_{i1}, b_{i2})} \left[\min_k (d_{ik}/b_{ik}) + \beta E_i z_j \right] \text{ s.t. } b_{i1} + b_{i2} = 2 \text{ and } c_{ik} \le \beta b_{ik} E_i z_j \text{ all } k.$$
(24)

By (20), $\phi_i \leq \min_k(d_{ik}/m_{ik}) + \beta E_i \phi_j$ so $G(\phi^\circ) \geq \phi^\circ$. Because $G(z) \geq G(z')$ for $z \geq z'$, the Tarski fixed-point theorem implies that G has a greatest fixed point z° and $z^\circ \geq \phi^\circ$. Let $\Delta_i = \min\{d_i, \beta E_i z_j^\circ \min_k(d_{ik}/c_{ik}^*)\}$ and let b_{ik}° denote the optimal b_{ik} for the problem in (24) when $z = z^\circ$. We claim $\min_k(d_{ik}/b_{ik}^\circ) = \Delta_i$, which is verified in the appendix. By the claim and $G_i(z^\circ) = z_i^\circ$, $z_i^\circ = \Delta_i + \beta E_i z_j^\circ$. By repeated substitution, $z_i^\circ = \Delta_i + \beta E_i z_j^\circ$ yields $\beta E_i z_j^\circ = \sum_{t\geq 1} \sum_j \beta^t \pi_{ij}(t) \Delta_j$. By constraints in (24), $\beta E_i z_j^\circ \geq c_i$ so $\beta E_i z_j^\circ = \max\{\beta E_i z_j^\circ, c_i\}$. Then by definitions of $\nu_i(\beta)$, δ_i , and z_i^* , we see that ν_i is $\beta E_i z_j^\circ$, δ_i is Δ_i , and $(z_i^*)_{i=1}^I$ is z° .

When *i* is the current state, Lemma 4 says that for the fixed-rate regime, x_{ik}^* is the maximal possible current value of currency *k* (which is also the maximal possible current stage-1 price of currency *k*) and $v_{ik}(\beta)$ is the maximal possible future value of currency *k*, and Lemma 5 says that for the fixed-rate regime, $2z_i^*$ is the maximal possible current value of two currencies (which is also the twice of the maximal possible common stage-1 price of both currencies) and $2\nu_i(\beta)$ is the maximal possible future value of two currencies. Notice that $\delta \leq d$; thus $z_i^* \leq 0.5\sum_k x_{ik}^*$, $\nu_i(\beta) \leq 0.5\sum_k v_{ik}(\beta)$, and the two inequalities become equalities iff $\delta = d$.

By the definition of $v_{ik}(\beta)$, $v_{ik}(0) = 0$, $v_{ik}(1) = \infty$, and $v'_{ik}(\beta) > 0$. By the definition of $\nu_i(\beta)$ and Lemma 5, $\nu_i(0) = 0$, $\nu_i(1) = \infty$, and $v_i(\beta)$ is strictly increasing and continuos in β . Thus we get two values of the discount factor, $\beta_x(i,k)$ and $\beta_z(i)$, well defined by

$$v_{ik}(\beta_x(i,k)) = c_{ik}^*,\tag{25}$$

$$\nu_i(\beta_z(i)) = c_i^*. \tag{26}$$

Let $\beta_x = \max_{(i,k)} \beta_x(i,k)$ and $\beta_z = \max_i \beta_z(i)$. By Lemma 4 and (22), the flexible-rate regime can cover the cost of producing y_{ik}^* all (i,k) only if $\beta \ge \beta_x$; by Lemma 5 and (23), the fixed-rate regime can do so only if $\beta \ge \beta_z$. These necessary conditions turn out to be sufficient.

Lemma 6 The efficient allocation y^* is supported by the flexible-rate regime iff $\beta \geq \beta_x$ and by the fixed-rate regime iff $\beta \geq \beta_z$.

Proof. We have already shown the "only-if" part. For the if" part, we first use the following problem to describe the meeting outcome $(y_{ik}(m, m'), \kappa_{ik}(m, m'), \iota_{ik}(m, m'))$ selected by the trading rule ξ in the supporting equilibrium.

Problem 1. Fix $m^* \in \mathbb{R}^{2I}_{++}$ and proceed by two steps. At step 1, determine a meeting outcome $(\bar{y}_{ik}(m,m'), \bar{\kappa}_{ik}(m,m'), 0)$ as follows: if $m_k \geq m^*_{ik}$ then let

$$(\bar{y}_{ik}(m,m'),\bar{\kappa}_{ik}(m,m')) = \arg\max_{q\geq 0,0\leq\kappa\leq m_k} [u_{ik}(q) - \beta\kappa E_i\phi_{jk}]$$
(27)

subject to $-c_{ik}(q) + \beta \kappa E_i \phi_{jk} \ge 0$; otherwise, let

$$(\bar{y}_{ik}(m,m'),\bar{\kappa}_{ik}(m,m')) = \arg\max_{q\geq 0,0\leq\kappa\leq m_k} [-c_{ik}(q) + \beta\kappa_{ik}E_i\phi_{jk}]$$
(28)

subject to $u_{ik}(q) - \beta \kappa E_i \phi_{jk} \geq 0$. At step 2, let the meeting outcome $\xi_{ik}(m, m')$ assigned by the rule ξ be

$$\xi_{ik}(m,m') = \arg \max_{q \ge 0, 0 \le \kappa \le m_k, 0 \le \iota \le m_l} \left[-c_{ik}(q) + \beta E_i(\kappa \phi_{jk} + \iota \phi_{jl}) \right]$$
(29)

subject to $u_{ik}(q) - \beta E_i(\kappa \phi_{jk} + \iota \phi_{jl}) \ge u_{ik}(\bar{y}_{ik}(m, m')) - \beta E_i \bar{\kappa}_{ik}(m, m') \phi_{jk}$.

For the flexible-rate regime, the supporting equilibrium (ϕ, ξ) has ϕ_{ik} equal to x_{ik}^* in Lemma 4 and ξ as the one in Problem 1 with $m_{ik}^* = 1$; for the fixed-rate regime, the supporting equilibrium (ϕ, ξ) has ϕ_i equal to z_i^* in Lemma 5 and ξ as the one in Problem 1 with $m_{ik}^* = c_{ik}^*/c_i^*$. To confirm, consider the flexible-rate regime and the argument for the fixed-rate regime is similar. Fix a country-k meeting in state i between a consumer who resides in country k' and carries $\eta(m_k)$ and a producer who carries m' = (0, 0). Let q denote the output and κ the payment of currency kin the meeting outcome assigned by ξ . If $m_k \ge 1$, then by (27), $c_{ik}(q) = \kappa v_{ik}(\beta)$ and $u'_{ik}(q) \ge c'_{ik}(q)$, strict only if $c^*_{ik} > m_k v_{ik}(\beta)$; by $\beta \ge \beta_x$, $v_{ik}(\beta) \ge c^*_{ik}$ so $q = y^*_{ik}$ and the consumer's payoff from the meeting outcome is $u^*_{ik} - c^*_{ik} + m_k v_{ik}(\beta) + A_{ik'}$ (see (2) for $A_{ik'}$). If $m_k < 1$ then by (28), the consumer's payoff from the meeting outcome is $m_k v_{ik}(\beta) + A_{ik'}$. Because the cost to carrying m_k into the meeting is $m_k \phi_{ik}$ and $\phi_{ik} > v_{ik}(\beta)$, it is optimal for the consumer to leave stage 1 with $m_k = 1$.

In the proof of Lemma 6, the outcome $\xi_{ik}(m, m')$ determined by the step-2 optimization (29) in Problem 1 is in the pairwise core because there is no restriction on which currency can be used in payments.⁷ This outcome maximizes the pro-

⁷Problem 1 does not represent an extensive game form with two rounds of alternating offers but it may be understood as a gradual bargaining problem (see O'Neill et al [29]). Following Hu et al

ducer's payoff conditional on not making the consumer worse off than the trade $(\bar{y}_{ik}(m, m'), \bar{\kappa}_{ik}(m, m'), 0)$ obtained from the step-1 optimization. Because of the restriction on the payment, the step-1 optimization turns the producer's home currency as the *right currency* for the meeting and, hence, endogenizes imperfect currency substitution (actually no substitution). Indeed, if the consumer does not carry the right currency, then the step-1 outcome is (0, 0, 0) so that the step-2 optimization gives the producer all surplus from trade; this scheme is borrowed from Zhu and Wallace [36].⁸ Conditional on that the consumer if he carries a sufficient amount of right currency (sufficiency is measured by m_{ik}^*) and to the producer otherwise; borrowed from Hu et al [11] and Hu and Rocheteau [12], this schemes encourages the consumer to spend a sufficient amount of real resources to acquire the right asset even when its rate of return is low.

When the two cutoff values β_x and β_z in Lemma 6 differ, one regime dominates another over a range of values of β (the fixed-rate regime dominates if $\beta_x > \beta_z$).

Proposition 3 Suppose (i) given π , α is not in a measure-zero set in \mathbb{R}^4 , and (ii) $u_{i1}^*/c_{i1}^* = u_{i2}^*/c_{i2}^*$ all i. Then $\beta_x > \beta_z$. In general, which regime dominates may depend on the aggregate shock and preferences.

Proof. By condition (ii), $d_i/c_i^* = d_{ik}/c_{ik}^*$ so $\max\{\nu_i(\beta), c_i^*\} \min_k(d_{ik}/c_{ik}^*) \ge d_i$. Then by definitions of δ_i and $\nu_i(\beta)$ in Lemma 5 and $v_{ik}(\beta)$ in Lemma 4, we see $\delta = d$ and $\nu_i(\beta) = 0.5 \sum_k v_{ik}(\beta)$ all *i*. Now fix *i* and let $\beta_x(i) = \max\{\beta_x(i, 1), \beta_x(i, 2)\}$. We claim that $\beta_x(i) \ge \beta_z(i)$ and $\beta_x(i) = \beta_z(i)$ only if $\beta_x(i, 1) = \beta_x(i, 2)$. As verified in the appendix, given π , $\beta_x(i, 1) \ne \beta_x(i, 2)$ when α is outside a measure-zero set in \mathbb{R}^4 . So by condition (i) and the claim, $\beta_x(i) > \beta_z(i)$. Because *i* is arbitrary, $\beta_x > \beta_z$. To verify the claim, suppose $\beta_z(i) \ge \beta_x(i)$. Then we have

$$c_i^* = \nu_i(\beta_z(i)) \ge \nu_i(\beta_x(i)) = 0.5 \sum_k v_{ik}(\beta_x(i)) \ge 0.5 \sum_k v_{ik}(\beta_x(i,k)) = c_i^*, \quad (30)$$

where the first equality uses the definition of $\beta_z(i)$ in (26), the first inequality uses strict monotonicity of $\nu_i(\beta)$ and it is strict if $\beta_z(i) > \beta_x(i)$, the second equality

^{[11],} one may adopt the game form used in Zhu [35] to implement $\xi_{ik}(m, m')$.

⁸One may obtain imperfect substitution of currencies by assuming that one currency differs from another in some fundamental aspect; for example, one currency is harder to counterfeit than another (see Gomis-Porqueras et al [8] and Zhang [34] for related models). Our purpose here is to examine which regime is better when thee is no currency substitution and when two currencies do not differ in any fundamental aspect; the Zhu and Wallace [36] scheme fits this purpose well.

uses $\nu_i(\beta) = 0.5 \sum_k v_{ik}(\beta)$, the second inequality uses $v'_{ik}(\beta) > 0$ and it is strict if $\beta_x(i,1) \neq \beta_x(i,2)$, and the last equality uses the definition of $\beta_x(i,k)$ in (25); thus the claim must be true.

Next, let us examine what may happen when condition (ii) fails by an example where I = 2, the aggregate shock is symmetric, and $\pi_{11} = 0.5$. By Lemma 4, $v_{ik}(\beta) = \beta \bar{d}_k/(1-\beta)$, where $\bar{d}_k = 0.5(d_{1k} + d_{2k})$. By symmetry, $\max_k c_{1k}^* = \max_k c_{2k}^*$ and $\bar{d}_k = d_1$. So by (25), $\max_k c_{1k}^* = \beta_x d_1/(1-\beta_x)$ or

$$(1-\beta_x)/\beta_x = \min_k (d_1/c_{1k}^*).$$

Also by symmetry, c_i^* , d_i , $\nu_i(\beta)$, and $\min_k(d_{ik}/c_{ik}^*)$ are all constant in *i*. So by Lemma 5, $\nu_i(\beta) = \beta \min\{d_i, \max\{\nu_i(\beta), c_i^*\} \min_k(d_{ik}/c_{ik}^*)\}/(1-\beta)$; then by (26), we have $c_1^* = \beta_z \min\{d_1, c_1^* \min_k(d_{1k}/c_{1k}^*)\}/(1-\beta_z)$. By $d_1/c_1^* \ge \min_k(d_{1k}/c_{1k}^*)$, this implies

$$(1 - \beta_z)/\beta_z = \min_k (d_{1k}/c_{1k}^*).$$

It follows that the sign of $\beta_x - \beta_z$ is the same as the sign of $\min_k (d_{1k}/c_{1k}^*) - \min_k (d_1/c_{1k}^*)$. Now set $u(q) = q - 0.5q^2$, c(q) = q, $\theta_{11} = \theta_{12} = 1$, and $\rho_{11} < \rho_{12} < 1$. As verified in the appendix, when $0.5 > \rho_{12} > \rho_{11}$, $\beta_x < \beta_z$; when $\rho_{12} > \rho_{11} > 0.5$, $\beta_x > \beta_z$.

To understand Proposition 3, it helps relate the term d_{ik} to the exogenous dividend in a version of the Lucas [22] asset-pricing model with risk neutrality. Specifically, at state i, d_{ik} is the dividend contributed by country k, d_i is the unintegrated worldwide dividend, and δ_i is the integrated worldwide dividend. In this connection, $v_{ik}(\beta)$ is solely determined by the dividend stream of country k, and the unified value $\nu_i(\beta)$ obtained by integrating the separated values $v_{i1}(\beta)$ and $v_{i2}(\beta)$ of two currencies is determined by the integrated worldwide dividend stream. Integration is costless if the integrated worldwide dividend is equal to the unintegrated worldwide dividend at all states (i.e., $\delta = d$) and is costly otherwise (i.e., $\delta \leq d$ and $\delta_j < d_j$ some j). Costless integration is ensured by condition (ii), a condition applicable to a class of familiar preferences (i.e., the functions u and c are power functions).

For the purpose of sustaining the efficient worldwide output, each country must rely on its own dividend stream to cover its production cost (i.e., $v_{ik}(\beta) \ge c_{ik}^*$) under the flexible-rate regime, while the two countries can share the integrated worldwide dividend stream to cover the worldwide production cost (i.e., $\nu_i(\beta) \ge c_i^*$) under the fixed-rate regime. Thus, by integrating $v_{i1}(\beta)$ and $v_{i2}(\beta)$ into $\nu_i(\beta)$, the fixed-rate regime promotes risk sharing between the two countries—it permits the worldwide efficiency to be sustained at state *i* in case that the dividend stream of country *k* is not sufficient (i.e., $v_{ik}(\beta) < c_{ik}^*$) but the integrated worldwide dividend stream is sufficient (i.e., $\nu_i(\beta) \ge c_i^*$).

When integration is costless, $\nu_i(\beta) = 0.5 \sum_k v_{ik}(\beta)$. Because existence of *i* with $v_{ik}(\beta) < c_{ik}^*$ and $0.5 \sum_k v_{ik}(\beta) \ge c_i^*$ is generic, the fixed-rate regime dominates thanks to the risk-sharing effect.

When integration is costly, there is value losing in that $\nu_i(\beta) < 0.5 \sum_k v_{ik}(\beta)$. To think of costly integration and value losing, let constraints in (20) and (21) be binding and $m'_{ik} = 0$; then (20) can be written as $\phi_{ik} = (u^*_{ik}/c^*_{ik})\beta E_i\phi_{jk}$. Refer to u^*_{ik}/c^*_{ik} as the country k's pairwise consumption-production condition at state i; this condition tells how much util gain from consumption can be obtained from per unit util cost of production in a country k's pairwise meeting at state i. Then $\phi_{ik} = (u^*_{ik}/c^*_{ik})\beta E_i\phi_{jk}$ says that people are willing to pay a higher price for currency k in the stage-1 market at a state if country k has a better pairwise condition at that state. But fixing exchange rates forces the stage-1 price of each currency to depend on the worse of the two conditions at the state. So when the two pairwise conditions differ at a state (i.e., $u^*_{j1}/c^*_{j1} \neq u^*_{j2}/c^*_{j2}$ some j), some country's dividend at that state is lost (i.e., $\delta_j < d_j$). In general equilibrium, the loss of the current dividend passes to the future value of currencies, resulting in $\nu_i(\beta) < 0.5 \sum_k v_{ik}(\beta)$.

With costly integration, which regime dominates is ambiguous. The example in the proof of Proposition 3 consists of ingredients which would make risk sharing the dominant factor—two countries are *ex ante* identical, one country in one state mirrors another country in another state, and the aggregate shock is transitory. But the value-losing effect can actually dominate even when the country-specific risk is small (i.e., $|\rho_{11} - \rho_{12}|$ is small).

5 Concluding remark

Both Proposition 2 and Proposition 3 find that neither regime is unambiguously better. This finding suggests that in general, neither regime would be the optimal way to manage exchange rates for worldwide production. But what is the optimal way? This calls for a formulation of a general exchange-rate system and analysis of such a system, a task that is left for the future research.

Appendix

The real-exchange rate in section-3.1 example

The real exchange rate at state i is determined as

$$rer_{i} = \frac{\phi_{i1}}{\phi_{i2}} \frac{(1 - v_{i1})(1/\phi_{i1}) + v_{i1}(1/y_{i1})}{(1 - v_{i2})(1/\phi_{i2}) + v_{i2}(1/y_{i2})}.$$
(31)

In (31), $(1-v_{ik})(1/\phi_{ik}) + v_{ik}(1/y_{ik})$ measures the *country-k price level* and $1-v_{ik}$ and v_{ik} are weights for the price of the stage-1 good in the unit of currency k and the price of the stage-2 good produced in country k in the unit of currency k, respectively. The weight v_{ik} is determined by the contribution of the stage-2 domestic output to total country-k output. Because there is no value added in the production of the stage-1 good, net output at stage 1 is zero and, hence, $v_{ik} = 1$. Applying this to (31) and using (5), we have $rer_i = u'_{i1}(y_{i1})/u'_{i2}(y_{i2})$.

As it turns out, persistence of the shock is a factor for the real exchange rate in the flexible-rate equilibrium. So let $(\pi_{11}, \pi_{12}) = (\mu, 1-\mu)$ and $\mu \ge 0.5$. This generalization does not affect any data for the fixed-rate equilibrium given in section 3.1. In the flexible-rate equilibrium, $(E_1\phi_{j1}, E_1\phi_{j2}) = (\mu\theta_{11} + (1-\mu)\theta_{12}, \mu\theta_{12} + (1-\mu)\theta_{11})$ and $(y_{11}, y_{12}) = (\sqrt{\beta E_1\phi_{j1}/\rho_{11}}, \sqrt{\beta E_1\phi_{j2}/\rho_{12}})$. Let $\delta_{\theta} = \theta_{11} - 1$ and $\delta_{\rho} = \rho_{11} - 1$. By a first order approximation, the variances of output and the real exchange rate in each country are $0.25[(1-2\mu)\delta_{\theta}+\delta_{\rho}]^2$ and $[(1+2\mu)\delta_{\theta}+\delta_{\rho}]^2$, respectively, in the flexible-rate equilibrium; they are $0.25(\delta_{\theta} + \delta_{\rho})^2$ and $(2\delta_{\theta} + \delta_{\rho})^2$ in the fixed-rate equilibrium. In each equilibrium, the autocorrelations of output and the real exchange rate in each country are $2\mu - 1$.

Proof of Lemma 1

We assume the following regularity condition:

Condition 1 $D_0 \neq \beta \pi_{ii} D_1$ all *i*, where $D_0 = c'(q^\circ) + q^\circ c''(q^\circ)$ and $D_1 = u'(q^\circ) + q^\circ u''(q^\circ)$.

The 4*I* equations in the main text can be written as $F_{ik}(\phi, y, \alpha) = 0$ all (i, k), where $F_{ik} = (F_{ik}^c, F_{ik}^p)$ and

$$F_{ik}^c(\phi, y, \alpha) = \phi_{ik} - y_{ik}\theta_{ik}u'(y_{ik}), \qquad (32)$$

$$F_{ik}^{p}(\phi, y, \alpha) = y_{ik}\rho_{ik}c'(y_{ik}) - \beta E_{i}\phi_{jk}; \qquad (33)$$

Now $\partial F_{ik}^c / \partial \phi_{ik} = 1$; $\partial F_{ik}^p / \partial \phi_{jk} = -\beta \pi_{ij}$ (all j), $\partial F_{ik}^c / \partial y_{ik} = -\theta_{ik} [u'(y_{ik}) + y_{ik}u''(y_{ik})]$, and $\partial F_{ik}^p / \partial y_{ik} = \rho_{ik} [c'(y_{ik}) + y_{ik}c''(y_{ik})]$; $\partial F_{ik}^c / \partial \phi_{jk}$, $\partial F_{ik}^c / \partial y_{jk}$, and $\partial F_{ik}^p / \partial y_{jk}$ vanishes if $j \neq i$. Hence, the Jacobian matrix of $(F_{1k}, ..., F_{Ik})$ with respect to (ϕ, y) evaluated at $(\phi, y, \alpha) = (\phi^{\circ}, y^{\circ}, 1)$ is

$$\partial F_{\phi yk} = \begin{bmatrix} \mathbf{I} & \vdots & -D_1 \mathbf{I} \\ \cdots & \cdots & \cdots \\ -\beta \mathbf{\Pi} & \vdots & D_0 \mathbf{I} \end{bmatrix}, \qquad (34)$$

where $\mathbf{\Pi} = (\pi_{ij})$ and \mathbf{I} is the $I \times I$ identity matrix. By its structure, the matrix in (34) is invertible if its *i*th and (i + I)th columns are linearly independent for $1 \leq i \leq I$. This is the case if the *i*th and (i + 1)th rows of these two columns constitute an invertible matrix. That, in turn, follows from Condition 1. Then by the implicit function theorem, there exists a unique $(\phi(\alpha), y(\alpha))$ for α in a neighborhood N of $1 \in \mathbb{R}^4$. That $\phi(\alpha)$ is unique is implied by (33).

Completion proof of Proposition 1

To begin with, let the mapping Φ on the neighborhood N of $1 \in \mathbb{R}^{4I}$ be defined by $\Phi(\alpha) = (\Phi_1(\alpha), \Phi_2(\alpha))$ and $\Phi_k(\alpha) = (\phi_{1k}(\alpha), ..., \phi_{Ik}(\alpha), y_{1k}(\alpha), ..., y_{Ik}(\alpha))$, where Nand $(\phi(\alpha), y(\alpha))$ are given at the end of the proof of Lemma 1. By the implicit function theorem, the Jacobian matrix $\partial \Phi_k$ of Φ_k evaluated at 1 is $\partial \Phi_k = -\partial F_{\phi yk}^{-1} \partial F_{\alpha k}$, where $\partial F_{\phi yk}^{-1}$ is the inverse of $\partial F_{\phi yk}$ in (34) and $\partial F_{\alpha k}$ is the Jacobian matrix of $(F_{1k}, ..., F_{Ik})$ for F_{ik} in (32) and (33) with respect to α evaluated at $(\phi, y, \alpha) = (\phi^{\circ}, y^{\circ}, 1)$. Because

$$\partial F_{\alpha k} = \begin{bmatrix} -q^{\circ}u'(q^{\circ})\mathbf{I} & \vdots & 0\\ \cdots & \cdots & \cdots\\ 0 & \vdots & q^{\circ}c'(q^{\circ})\mathbf{I} \end{bmatrix}$$

 $\partial \Phi_k$ is invertible and so is the Jacobian matrix $\partial \Phi$ of Φ evaluated at 1. Next define the mapping $(\phi, y) \mapsto \Omega(\phi, y)$ from $S' = \{(\phi, y) \in R^{4I}_{++} : \phi_{i1}\phi_{12} = \phi_{11}\phi_{i2}\}$ to R^{I-1} by $\Omega_i(\phi, y) = \phi_{i1}\phi_{12} - \phi_{11}\phi_{i2}$ for $2 \leq i \leq I$. Apparently, the Jacobian matrix $\partial\Omega$ of $\Omega(\phi, y)$ has full rank I - 1 ($\partial\Omega_i/\partial\phi_{i1} = \phi_{12}$, $\partial\Omega_i/\partial\phi_{i2} = -\phi_{11}$, $\partial\Omega_i/\partial\phi_{jk}$ vanishes if $j \neq i$, and $\partial\Omega_i/\partial y_{jk}$ vanishes all (i, k)). Finally, note that the set S is the zero set of the composition mapping $\Omega \cdot \Phi$ from N to \mathbb{R}^{I-1} . Because the product of $\partial\Omega$ and $\partial\Phi$ has the full rank I - 1, $0 \in \mathbb{R}^{I-1}$ is a regular value of $\Omega \cdot \Phi$.

Proof of Lemma 2

For the first assertion in the lemma, let $\phi_j = \phi_{j1} = \phi_{j2}$. First consider T_{hi} . The perunit price of each currency is ϕ_i/a_h on the current market and $1-a_h$ units of currencies are withdrawn, where $a_h = s_{h1}\gamma_{h1} + s_{h2}\gamma_{h2}$ (see (1)). So $T_{hi} = (\phi_i/a_h)(1-a_h)$. By $a_h\zeta_h\phi_h = E_h\phi_j$, $T_{hi} = \varphi_i(\zeta_h\phi_h/E_h\phi_j - 1)$ and the expression in (13) follows from $m_{ik}\phi_i = m_{ik}\phi_{ik} = \varphi_{ik}$ (see (5)), $m_{i1} + m_{i2} = 2$, and $\varphi_{i1} + \varphi_{i2} = \varphi_i$. Next consider T'_{hi} . The per-unit price of currency k is ϕ'_{ik}/γ'_{hk} and $0.5(1 - \gamma'_{hk})$ units of currency k are withdrawn. So $T'_{hi} = 0.5\sum_k(\phi'_{ik}/\gamma'_{hk})(1 - \gamma'_{hk})$. By (11), $T'_{hi} =$ $0.5\sum_k[\phi'_{ik}\phi'_{hk}\zeta_h/E_h\phi'_{jk} - \phi'_{ik}]$ and by (10), $T'_{hi} = 0.5\zeta_h\sum_k\phi_i m_{ik}\phi_h m_{hk}/E_hm_{jk}\phi_j - \phi_i$. Then (13) follows from $m_{ik}\phi_i = m_{ik}\phi_{ik} = \varphi_{ik}$.

For the second assertion, note that $A_{ik} = 0.5\beta E_i [-2\tau_{ijk} + \Delta_{jk} + 2A_{jk} - \Lambda_{jk}]$, where $\Delta_{jk} = (1-\lambda)u_{jk}(y_{jk}^*) + \eta u_{jl}(y_{jl}^*) - c_{jk}(y_{jk}^*)$ (λ is the probability for a consumer to be a tourist) and $\Lambda_{jk} = \lambda y_{jk}^* u'_{jk}(y_{jk}^*) + (1-\lambda)y_{jl}^* u'_{jl}(y_{jl}^*) - y_{jk}^* c'_{jk}(y_{jk}^*)$ with $l \neq k$. Let $\Delta_j = \sum_k [u'_{jk}(y_{jk}^*) - c'_{jk}(y_{jk}^*)]$. Using $E_i T_{ij} = 0.5\beta^{-1}\varphi_i - 0.5E_i\varphi_j$ and $u'_{jk}(y_{jk}^*) = c'_{jk}(y_{jk}^*)$, then $A_i \equiv A_{i1} + A_{i2} = -0.5\varphi_i + 0.5\beta E_i(\Delta_j + \varphi_j + 2A_j)$. By repeated substitution, $A_i = -0.5\varphi_i + 0.5\sum_{t\geq 1}\sum_j \beta^t \pi_{ij}(t)\Delta_j$. Now (14) follows from applying this A_i to (13) and $\zeta_h = \beta^{-1}$ to (12).

Completion of proof of Proposition 2

We first verify non-genericity of $\varphi_{i1}E_h\varphi_{j2} = \varphi_{i2}E_h\varphi_{j1}$. Fix (i, h) and define the mapping $\alpha \mapsto \Gamma(\alpha)$ by $\Gamma(\alpha) = y_{i1}c'_{i1}(y_{i1})E_hy_{j2}c'_{j2}(y_{j2}) - y_{i2}c'_{i2}(y_{i2})E_hy_{j1}c'_{j1}(y_{j1})$, where y_{jk} is an implicit function of α determined by $c'_{jk}(y_{jk}) = u'_{jk}(y_{jk})$. When $h \neq i$, $\partial \Gamma/\partial \rho_{h1} = -\varphi_{i2}\pi_{hh}[c'_{h1}(y_{h1}) + y_{h1}c''_{h1}(y_{h1})]\partial y_{h1}/\partial \rho_{h1}$. When h = i and $h' \neq h$, $\partial \Gamma/\partial \rho_{h'1} = -\varphi_{i2}\pi_{hh'}[c'_{h'1}(y_{h'1}) + y_{h'1}c''_{h'1}(y_{h'1})]\partial y_{h'1}/\partial \rho_{h'1}$. Therefore, it follows from $\partial y_{j1}/\partial \rho_{j1} = -c'(y_{j1})[c''_{j1}(y_{j1}) - u''_{j1}(y_{j1})]^{-1}$ that the Jacobian of Γ evaluated at any α has full rank. So the dimension of the zero set of Γ is 4I - 1.

Now we give an example with $\beta_{fix} > \beta_{flex}$. Let c(y) = y and let u be the same as in the example in section 3.1. Let I = 3. Let $(\pi_{i1}, \pi_{i2}, \pi_{i3}) = (\mu_1\psi_1, \mu_1\psi_2, \mu_2)$ for i = 1, 2 and $(\pi_{i1}, \pi_{i2}, \pi_{i3}) = (\mu_2/2, \mu_2/2, \mu_1)$, where $\psi_1 + \psi_2 = \mu_1 + \mu_2 = 1$. Let (i) $\theta_{31} = \theta_{11} > \theta_{21}$, (ii) $\theta_{32} > \theta_{12} > \theta_{22}$, and (iii) $\theta_{11}/\theta_{12} > E_1\theta_{j1}/E_1\theta_{j2} > \theta_{31}/\theta_{32}$. By (iii) and (15), $g_{13} > f_{13}$. A simple way to ensure that (1, 3) attains β_{flex} is to vary μ_2 and ρ_{ik} . To see how this work, first note that given (i) and (ii), $f_{13} \ge f_{hi}$ for $h, i \in \{1, 2\}$ (note $E_1\theta_{ik} = E_2\theta_{ik}$). Using (i) and (ii) once more, we have $(1, 3) = \operatorname{argmax} f_{hi}$ if $f_{13} > f_{33}$. When μ_2 is close to 0, $\theta_{1k}/E_1\theta_{ik} > \theta_{3k}/E_3\theta_{ik}$ so (i) and (ii) ensure $f_{13} > f_{33}$. Note that linearity of c implies $A_{ik} = 0$ so by adjusting ρ_{ik} , we can further ensure $V_1(\beta) = V_2(\beta) > V_3(\beta)$.

Completion of proof of Lemma 5

For existence of $\nu(\beta)$, let $\underline{\nu} = (\underline{\nu}_i)_{i=1}^I$ have $\underline{\nu}_i = \beta \min_j [c_j^* \min_k (d_{jk}/c_{jk}^*)]/(1-\beta)$ all i. Define a mapping $\nu \mapsto F(\nu)$ from the set $\{\nu = (\nu_i)_{i=1}^I : \nu \ge \underline{\nu}\}$ to \mathbb{R}^I by

$$F_i(\nu) = \sum_{t \ge 1} \sum_{j=1} \beta^t \pi_{ij}(t) \min\{d_j, \max\{\nu_j, c_j^*\} \times \min_k (d_{jk}/c_{jk}^*)\}.$$
 (35)

Note that for $\nu \geq \nu'$, $F(\nu) \geq F(\nu') \geq F(\underline{\nu})$. Then by the Tarski fixed-point theorem, F has a greatest fixed point, which is $\nu(\beta)$. For monotonicity of $\nu(.)$, let $\beta_2 > \beta_1$ and replace $\underline{\nu}$ with $\nu(\beta_1)$ in the domain of F and set $\beta = \beta_2$ in (35); then applying the Tarski fixed-point theorem once more, we get $\nu(\beta_2) > \nu(\beta_1)$. For continuity of $\nu(.)$, it suffices to consider the case that $\nu_j(\beta) = c_j^*$ for some j. For this case, it is clear that $\nu(\beta_n) \to \nu(\beta)$ as $\beta_n \uparrow \beta$. Setting $F_i(\nu) = \nu_i$, (35) can be written as linear equations $P(\beta)\nu = Q(\beta)$ in ν . Because the greatest fixed point of F is also the greatest solution to the linear equations $P(\beta)\nu = Q(\beta)$, $P(\beta)$ is invertible (otherwise there is a continuum of solutions and none of which can be the greatest). Therefore, $[P(\beta)]^{-1}Q(\beta) = \nu(\beta)$. When $\beta_n \downarrow \beta$, $\nu(\beta_n) = [P(\beta_n)]^{-1}Q(\beta_n) \to [P(\beta)]^{-1}Q(\beta)$.

For the claim $\min_k(d_{ik}/b_{ik}^\circ) = \Delta_i$, note either $b_{ik}^\circ \beta E_i z_j^\circ > c_{ik}^*$ for both k or not. If the former, then using $b_{ik}^\circ = b_{ik}^* \equiv d_{ik}/d_i$ and $b_{ik}^\circ \beta E_i z_j^\circ > c_{ik}^*$, we have $\beta E_i z_j^\circ(d_{ik}/c_{ik}^*) > d_i = d_{ik}/b_{ik}^\circ$, confirming the claim. So suppose without loss of generality that $b_{i1}^\circ \beta E_i z_j^\circ = c_{i1}^*$. Then $\beta E_i z_j^\circ(d_{i1}/c_{i1}^*) = d_{i1}/b_{i1}^\circ < d_{i1}/b_{i1}^* = d_i$ and $\beta E_i z_j^\circ(d_{i2}/c_{i2}^*) \ge d_{i2}/b_{i2}^\circ \ge d_{i2}/b_{i2}^* = d_i$, again confirming the claim.

Completion of proof of Proposition 3

Here we first verify non-genericity of $\beta_x(i,1) = \beta_x(i,2)$. Fix *i* and define the mapping $\alpha \mapsto \Gamma_k(\alpha)$ by $\Gamma_k(\alpha) = \sum_{t\geq 1} \sum_j \beta^t \pi_{ij}(t) d_{jk} - c_{ik}^*$, where y_{jk} is an implicit function of α determined by $c'_{jk}(y_{jk}) = u'_{jk}(y_{jk})$. Let $\Gamma(\alpha) = \Gamma_1(\alpha) - \Gamma_2(\alpha)$. Fix $j \neq i$. Using $\partial \Gamma / \partial \theta_{j1} = \sum_{t\geq 1} \sum_j \beta^t \pi_{ij}(t) [u(y_{j1}) + u'_{j1}(y_{j1}) - c'_{j1}(y_{j1})] \partial y_{j1} / \partial \theta_{j1}$, we have $\partial \Gamma / \partial \theta_{j1} = \sum_{t\geq 1} \sum_j \beta^t \pi_{ij}(t) u(y_{j1}) \partial y_{j1} / \partial \theta_{j1}$. Therefore, it follows from $\partial y_{j1} / \partial \theta_{j1} = u'(y_{j1}) [c''_{j1}(y_{j1}) - u''_{j1}(y_{j1})]^{-1}$ that the Jacobian of Γ evaluated at any α has full rank. So the dimension of the zero set of Γ is 4I - 1. Next we verify the assertion pertaining to the example. Using $y_{1k}^* = 1 - \rho_{1k}$, we have $d_{1k} = 0.5(1 - \rho_{1k})^2$, $c_{1k}^* = \rho_{1k}(1 - \rho_{1k})$, $d_{1k}/c_{1k}^* = 0.5(1/\rho_{1k} - 1)$, and $d_1 = 0.25[(1 - \rho_{11})^2 + (1 - \rho_{12})^2]$. So $d_{11} > d_1 > d_{12}$ and $d_{11}/c_{11}^* > d_{12}/c_{12}^* = \min_k(d_{1k}/c_{1k}^*)$. When $0.5 > \rho_{12} > \rho_{11}$, $c_{12}^* > c_{11}^*$ so $\min_k(d_1/c_{1k}^*) = d_1/c_{12}^*$. Thus $\beta_x < \beta_z$.

When $\rho_{12} > \rho_{11} > 0.5$, $c_{11}^* > c_{12}^*$ so $\min_k(d_1/c_{1k}^*) = d_1/c_{11}^*$; moreover, $d_1/c_{11}^* > d_{12}/c_{12}^*$ iff $(1-\rho_{11})^2 + (1-\rho_{12})^2 > \frac{2\rho_{11}(1-\rho_{11})(1-\rho_{12})}{\rho_{12}}$. Set $\varsigma(a) = (1-a)^2 + (1-\rho_{12})^2 - \frac{2a(1-a)(1-\rho_{12})}{\rho_{12}}$ for $a \le \rho_{12}$. Now $d_1/c_{11}^* > d_{12}/c_{12}^*$ follows from $\varsigma(\rho_{12}) = 0$ and $\varsigma'(a) < 0$. Thus $\beta_x > \beta_z$.

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