Debasements and Small Coins: An Untold Story of Commodity Money

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Abstract

This paper draws quantitative implications for some historical coinage issues from an existing formulation of a theory that explains the society's demand for multiple denominations. The model is parameterized to match some key monetary characteristics in late medieval England. Inconvenience for an agent due to a shortage of a type of coin is measured by the difference between his welfare given the shortage and his welfare in a hypothetical scenario that the mint suddenly eliminates the shortage. A small coin has a more substantial role than being small change. Because of this role, a shortage of small coins is highly inconvenient for poor people and, the inconvenience may extend to all people when commerce advances. A debasement may effectively supply substitutes to small coins in shortage. Large increase in the minting volume, cocirculation of old and new coins, and circulation by weight, critical facts constituting the debasement puzzle, emerge in the equilibrium path that follows the debasement.

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Key Words: The debasement puzzle; Gresham's Law; Medieval coinage; Commodity money; Coinage; Shortages of small coins

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1 Introduction

Debasements of coins were not rare in medieval Europe. When a type of coin was debased, i.e., its content of precious metal was reduced, a person could take bullion or old coins of the type to a mint in exchange for new coins. Rolnick, Velde, and Weber [15] find that a debasement tended to induce unusually large minting volumes and, at least some of the time following the debasement, old and new coins cocirculated by weight; they refer to their findings as the debasement puzzle because people did not receive any additional inducement to bring old coins to the mint (people actually paid fees for minting coins). Interestingly, debasements were often considered and sometimes implemented following the public complaints about inconvenience caused by shortages of small coins. Such complaints were widely recorded, motivating a view that the small-coin provision is a big problem for commodity money (see Cipolla [3], Redish [12], and Sargent and Velde [18]).

How should we measure a person's inconvenience due to a shortage of some coins? By what mechanism, would a debasement alleviate the shortage? Is the small-coin provision really a big problem for commodity money? If so, why? These issues and the debasement puzzle are the focus of our paper. Our starting point is a folk theory of the society's demand for multiple monetary objects. The theory consists of three ingredients: a wide range of transaction values, a burden of carrying a bulk of monetary objects, and indivisibility of monetary objects. To elaborate, suppose there is only one type of coin. If that coin facilitates all transactions, then high-value transactions may require many coins; but if high-value transactions only require a few coins, then even one coin may be too big for low-value transactions.

We build our work on the formalization of the folk theory provided by Lee, Wallace, and Zhu [9]. We measure the inconvenience for an individual agent when a type of coin is not supplied by the mint as the percentage change of his welfare given the complete shortage of the coin and his welfare in a hypothetical scenario that the coin is suddenly supplied by the mint. Probably, when a person in history complained about a shortage, he got a sense of inconvenience from comparing his real experience with his experience in an analogous hypothetical scenario. In the model, the hypothetical scenario is an unanticipated shock to the coinage structure. This approach naturally extends to debasement: the mint's sudden supply of the halfpenny is equivalent to debasing the penny by 50% while supplying the old penny by a different name. We parameterize the model to match key monetary characteristics of England in the fifteenth century. During this period, per capita holdings of silver in money varied but 35 grams may be a useful reference. Pennies (1d) were the mostly used coins; silver per penny declined over time but 1 gram is a good reference. The public complained about shortages of halfpennies (1/2d) and farthings (1/4d).

When a shock adds a small coin (e.g., the halfpenny) into the parameterized model, the small coin has a role more substantial than the small-change role emphasized by the folk theory.[1] With the small coin, each agent can smooth his consumption of goods purchased with money by way of spreading his purchasing power previously contained in 1 coin into 2 coins, say. The benefit is great for a poor agent even if he spends money once once a month. Provided that commerce is sufficiently advanced, i.e., monetary transactions are frequent enough, all agents are better off from an addition of a coin smaller than the farthing. Remarkably, the significant effects of consumption smoothing can be consistent with that coins such as groats (4d) and half groats (2d) are heavily used in transactions before the small coin is added. Debasing the penny by 50% has the similar welfare effect as adding the halfpenny. When a shock adds the sixpence (a large coin) or debases the shilling (12d) by 50%, the change in each agent's welfare is negligible. New coins, regardless of being large or small and regardless of being added or coming from a debasement, draw agents to the mint, and cocirculate with old coins by weight.

Why did a society make coins with precious metal? Plausibly, precious metal was a commitment device to prevent over-issuance of money. How costly was that device? There was an opportunity cost (see Sargent and Wallace [19] and Velde and Weber [24]). But what else? As noted by Redish [12], there is a practical lower bound on the metal content in a coin, for a low-fineness coin is easy to counterfeit and a high-fineness but low-content coin is too small to carry. In fact, a coin like the farthing is largely impractical—the weight of a high-fineness farthing is around 0.4 grams while the weight of a modern U.S. cent is 2.5 grams. By our study, then, the small-coin provision would impose a significant cost on the society. Given its significance, the cost may explain the experimentation with a variety of imperfect substitutes to full-bodied small coins before the society found an alternative commitment device,[2] and it may contribute to the final triumph of fiat money after.[3]

Debasement and shortages of small coins have drawn a fair amount of attention through the influential monograph of Sargent and Velde [18], *The Big Problem of Small Change*. Sargent and Velde [18] adapt the cash-in-advance model of Lucas and Stokey [11] by replacing cash and credit goods with penny and dollar goods: penny goods can only be bought with pennies (small coins) while dollar goods can be bought with dollars (large coins) and pennies; a shortage of pennies is identified with a binding penny-in-advance constraint, occurring when pennies depreciate relative to dollars.[4]

[1] Halfpennies were not small in medieval England. In 1490s, a whole pig would cost 33 pennies and one penny could buy 3.73 kg salt, 3.56 kg wheat, 1.20 kg cheese, or 4.35 kg wool; see Farmer [4, Tables 4, 7].

[2] The imperfect substitutes included billion coins, copper coins, pieces cut from coins, foreign coins with less metal content, etc.; see Redish [12, ch 4] for problems with billion coins and copper coins. The standard formula for the small-coin provision prescribes to issue token coins convertible to precious metal; see Cipolla [3]. But convertibility needs commitment.

[3] When the state commitment was somehow in place, the society adopted presumably convertible token coins (as prescribed by the standard formula) and notes in large denominations. The presumed convertibility finally phased out but the state commitment somehow keeps over-issuance in check.

[4] As noted by Wallace [26], users of the Lucas-Stokey model usually do not interpret a binding

In the Sargent-Velde model, a shortage is a demand-side problem; debasing the penny alleviates the shortage because the assumed circulation-by-tale enhances the agent's incentive to hold new pennies; and if a society finances the mint's operation, the small-coin problem is resolved because the zero minting fees eliminate all non-steady-state equilibria. In our model, a shortage is a supply-side problem; debasing the penny alleviates the shortage because new and old pennies circulate by weight and new pennies are smaller than old; and even if a society finances the mint's operation, the small-coin problem may not be resolved.

The rest of the paper is organized as follows. We set up the basic model in section 2. Quantitative results are presented in sections 3 and 4. We discuss our model and results and some other related literature in section 5. Section 6 concludes.

2 The basic model

Time is discrete, dated as $t \geq 0$. There is a unit measure of infinitely lived agents. There are two stages per period. At the start of the first stage of period t, each agent knows his type at the period—he becomes a buyer or a seller with equal chance. Then agents visit a mint that produces monetary items, referred to as coins, from a durable commodity, called silver. Silver has a fixed stock M; it can also be costlessly converted into and back from a product, called jewelry. There are K types of coins and a unit of coin k contains $m_k > 0$ units of silver, $1 \leq k \leq K$. A unit of jewelry contains m_0 units of silver. Agents choose their wealth portfolios in silver at the mint by the way described below. There is an exogenous upper bound B on each agent's silver wealth. At the second stage, agents carry coins into a decentralized market where each buyer is randomly matched with a seller. In each pairwise meeting, the seller can produce a perishable good that can only be consumed by the buyer. Trading histories are private information, ruling out credits between the two agents. In the meeting, each agent's wealth portfolio is observed by his meeting partner and the buyer makes a take-it-or-leave-it offer.

Let $Y_t = \prod_{k=0}^{K}\{0, 1, ..., B/m_k\}$ so $y = (y_0, ..., y_K) \in Y_t$ represents an agent's generic portfolio of wealth in silver at period t, meaning that the agent holds y_0 units of jewelry and y_k units of coin k, $k \geq 1$. Coins may exist at the start of period 0; that is, $m_0\pi_0(y_0, 0, ..., 0)$ may be less than M, where π_0 is the distribution of wealth portfolios in silver among agents at the start of period 0. If the agent visits the mint with $y \in Y_t$, he can choose a portfolio from the set

$$\Gamma_t(y) = \{y' \in Y_t : m \cdot y' = m \cdot y\}, \tag{1}$$

where $m = (m_0, ..., m_K)$. Here and below, $a \cdot b$ denotes the inner product of vectors a and b. If the agent ends with y' at stage 1 and if he consumes $q_b \geq 0$ (when he is a buyer) and produces $q_s \geq 0$ (when he is a seller) at stage 2, then his realized utility

cash-in-advance constraint as a shortage of cash.

at period t is

$$u(q_b) - q_s + v(m_0 y_0') - \gamma \cdot y'. \tag{2}$$

Here $\gamma = (\gamma_0, ..., \gamma_K)$, $\gamma_0 = 0$, and $\gamma_k = \gamma_C > 0$ is the disutility to carry a unit of coin k to the decentralized market; the utility functions u and v satisfy $u', v' > 0$, $u'' < 0, v'' \leq 0$, $v(0) = u(0) = 0$, and $u'(0) = \infty$.[5] Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$.

There may be an unanticipated shock to the coinage structure $(m_1, ..., m_K)$. The shock is either a *structure* shock that adds some types of coins into the pre-shock coinage structure or a *debasement* shock that reduces silver content in each of some $J \leq K$ types of coins in the pre-shock coinage structure. The shock is introduced at the start of period 0 (at that time agents only hold coins in the pre-shock coinage structure). The debasement is represented by a one-to-one mapping $j \mapsto d(j)$ such that if $k = d(j)$ for some $j \in \{1, ...J\}$, then coin k is debased. If coin k is debased, the pre-shock coin is called *old coin k* and the post-shock coin is called *new coin k*; the mint does not provide old coin k any more; and old coins can be held up to period $\hat{t} < \infty$ but must be melted at period $\hat{t}$.[6]

Next we turn to the equilibrium conditions. In the post-shock economy, let Y_t, m, $\Gamma_t(y)$, and γ be defined the same way as in the pre-shock economy for a distinct K following the structure shock and for a distinct $(m_1, ..., m_K)$ following the debasement shock when $t \geq \hat{t}$. Following the debasement shock when $t < \hat{t}$, let m_k^o denote the amount of silver per old coin k, $Y_t = \prod_{k=0}^{K} \{0, 1, ..., B/m_k\} \times \prod_{j=1}^{J} \{0, 1, ..., B/m_{d(j)}^o\}$, $m = (m_0, ..., m_K, m_{d(1)}^o ..., m_{d(J)}^o)$,

$$\Gamma_t(y) = \{y' \in Y_t : m \cdot y' = m \cdot y, \; y_{d(j)}^{o\prime} \leq y_{d(j)}^o\}, \tag{3}$$

and $\gamma = (\gamma_0, ..., \gamma_K, \gamma_{d(1)}^o, ..., \gamma_{d(J)}^o)$ with $\gamma_{d(j)}^o = \gamma_C$ all $j \geq 1$. The equilibrium conditions are described by a same set of constructs for the pre-shock and post-shock economies, with the understanding that the suitable Y_t, m, $\Gamma_t(y)$, and γ are applied.

For each period t, the set of constructs consists of three probability measures on Y_t, denoted π_t, θ_t^b, and θ_t^s, and three value functions on Y_t, denoted w_t, h_t^b, and h_t^s. Here $\pi_t(y)$ is the fraction of and $w_t(y)$ is the value for agents holding the wealth portfolio y before agents know their period-t types; $\theta_t^a(y)$ is the fraction of and $h_t^a(y)$ is the value for buyers (sellers, resp.) holding y right after visiting the mint at t when $a = b$ ($a = s$, resp.) In terms of h_t^a, the portfolio-choice problem for an agent holding y at the mint can be expressed as

$$g(y, h_t^a) = \max_{y' \in \Gamma_t(y)} h_t^a(y') + v(m_0 y_0'), \; a \in \{b, s\}. \tag{4}$$

[5] In this setup, we assume away the minting fees. The zero minting fees do not eliminate the debasement puzzle because given $\gamma_C > 0$, it still incurs an extra cost for an agent to melt one penny in exchange for two halfpennies.

Also, we follow Velde and Weber [24] by assuming that.silver yields direct utility only when it is used in jewelry. The use of silver in jewelry captures the idea that the money stock is not constant due to hoarding, international flow, and industry use of metal.

[6] A finite $\hat{t}$ is a simple way to capture that in history, old coins did eventually disappear for a variety of reasons (lost, deteriorated, etc) not considered in our model.

In terms of w_{t+1}, the trade in a pairwise meeting between a buyer with y_b and a seller with y_s solves the maximization problem

$$f(y_b, y_s, w_{t+1}) = \max_{(q,\iota)} u(q) + \beta w_{t+1}(y_b - \iota) \tag{5}$$

subject to $-q + \beta w_{t+1}(y_s + \iota) \geq \beta w_{t+1}(y_s)$ and $\iota \in L(y_b, y_s)$, where

$$L(y_b, y_s) = \{\iota \in Y_t : \iota = \iota_b - \iota_s, \iota_b, \iota_s \in Y_t, \iota_{b,0} = \iota_{s,0}, \tag{6}$$
$$\text{and } \forall k \geq 1,\ \iota_{b,k} \leq y_{b,k}, \iota_{s,k} \leq y_{s,k}\}$$

is the set of feasible coin transfers between the buyer and the seller. Given h_t^b and h_t^s, the function w_t satisfies

$$w_t(y) = 0.5g(y, h_t^b) + 0.5g(y, h_t^s). \tag{7}$$

As implied by the maximization problem in (5), the function h_t^s satisfies

$$h_t^s(y) = \beta w_{t+1}(y) - \gamma \cdot y. \tag{8}$$

Given w_{t+1} and θ_t^s, the function h_t^b satisfies

$$h_t^b(y) = \sum_{y'} \theta_t^s(y') f(y, y', w_{t+1}) - \gamma \cdot y. \tag{9}$$

Given π_t, the measure θ_t^a satisfies

$$\theta_t^a(y') = \sum_{y} \pi_t(y) \lambda_1^a(y'; y),\ a \in \{b, s\}, \tag{10}$$

for some $\lambda_1^a(.; y) \in \Lambda_1[y, h_t^a]$, where $\Lambda_1[y, h_t^a]$ is the set of measures that represent all randomizations over the optimal portfolios for the maximization problem in (4). Given θ_t^b and θ_t^s, the measure π_{t+1} satisfies

$$\pi_{t+1}(y) = \sum_{(y_b, y_s)} \theta_t^b(y_b)\theta_t^s(y_s)[\lambda_2(y; y_b, y_s) + \lambda_2(y_b - y + y_s; y_b, y_s)] \tag{11}$$

for some $\lambda_2(.; y_b, y_s) \in \Lambda_2[y_b, y_s, w_{t+1}]$, where $\Lambda_2[y_b, y_s, w_{t+1}]$ is the set of measures that represent all randomizations over the optimal transfers of coins for the maximization problem in (5) and $\lambda(y)$ is the proportion of buyers with y_b who leave with y after meeting sellers with y_s.

Definition 1 *In each of the pre-shock and post-shock economies, a monetary equilibrium is a sequence* $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ *that satisfies (4)-(11) all* t *and* $\sum_{\{y \in Y_t : y_k = 0, k \geq 1\}} m_0[\theta_t^b(y) + \theta_t^s(y)] < 2M$ *some* t *for a given* π_0 *and for the applicable* $(Y_t, m, \Gamma_t(y), \gamma)$*; a steady state is a tuple* $(w, \theta^b, \theta^s, \pi)$ *such that* $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ *with* $(w_t, \theta_t^b, \theta_t^s, \pi_t) = (w, \theta^b, \theta^s, \pi)$ *all* t *is a monetary equilibrium.*

For existence, let $m_* = \min_{k \geq 1} m_k$ and we maintain a simple sufficient condition

$$\frac{B - m_* - 0.5M}{B - m_*} u[\frac{\beta(v(B) - v(B - m_*))}{1 - \beta}] > v(B) + \frac{\beta}{1 - \beta}[v(B) - v(B - m_*)] + \gamma_C, \tag{12}$$

saying that the upper bound on silver wealth is not too strict, the smallest coin is not too large, the cost to carrying coins is not too great, and the utility of jewelry is much limited.

Proposition 1 *In each of the pre-shock and post-shock economies, there exists a monetary equilibrium for a given* π_0 *and there exists a monetary steady state.*

Proof. See the appendix. ■

3 Quantitative results

To conduct quantitative analysis, we set $M = 35$ and $m_0 = 60$ and let the baseline coinage structure be $(m_1, m_2, m_3, m_4) = (12, 4, 2, 1)$. These parameters are meant to approximate the monetary characteristics of England in the fifteenth century. One unit of silver in the model corresponds to 1 gram. As one penny contained around 1 gram of silver in the fifteenth century, coins 1 to 4 represent the shilling, groat, half groat, and penny, respectively; and $m_0 = 60$ is about 2 times of *troy ounce* (31 grams), the most common and smallest measure of precious metal in medieval England. It turns out that with $m_0 = 60$ and $M = 35$, agents hold most of the silver in coins. So the per capita silver in money in the model falls in the mid of the estimated range for England in the fifteenth century (see Allen [1, p. 607]). We set $B = 3M$. This upper bound on wealth in silver is not restrictive in that it is reached by a negligible measure of agents.[7]

Some studies suggest that medieval people had a lower discount factor than modern people (see Kimball [7]). So we set the annual discount rate at 10% and the discount factor is $\beta = 0.9^{1/F}$ when people have F rounds of pairwise meetings per year. We use $F = 24$ as the baseline value. For objects in (2), we set $u(x) = x^{1-\sigma}/(1-\sigma)$ and $\sigma = 0.5$, $v(x) = \varepsilon x/F$ and $\varepsilon = 0.01$, and $\gamma_C = 10^{-5}$. There is no obvious reference to pin down γ_C and ε but we prefer smaller values to larger. Given $F = 24$, roughly, $\gamma_C = 10^{-5}$ is equivalent to 0.005% of the steady-state per capita consumption, and $\varepsilon = 0.01$ suggests one additional unit of silver in jewelry yields a utility equivalent to 0.02% of the steady-state per capita consumption. The main patterns of the presented results hold when σ varies from 0.5 to 1, ε varies from 0.005 to 0.05, and γ_C varies from 10^{-4} to 10^{-6}, and when v has some strict curvature.

Given a shock, we compute a monetary steady state $(\tilde{w}, \tilde{\theta}^b, \tilde{\theta}^s, \tilde{\pi})$ in the pre-shock economy, a monetary steady state $(w, \theta^b, \theta^s, \pi)$ in the post-shock economy, and a monetary equilibrium $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ in the post-shock economy that starts with $\pi_0 = \tilde{\pi}$ and converges to $(w, \theta^b, \theta^s, \pi)$. Proposition 1 does not tell uniqueness of the monetary steady state in either economy; but for the given parameter values, our algorithm always converges to the same steady state from a variety of initial conditions. Moreover, Proposition 1 does not assure existence of a post-shock monetary equilibrium $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ with the desired property of $\lim_{t\to\infty}(w_t, \theta_t^b, \theta_t^s, \pi_t) = (w, \theta^b, \theta^s, \pi)$. In computation, our algorithm approximates that property by letting $w_T = w$ for a sufficient large T (often $T = 500$ serves the purpose). Details of all algorithms are given in the appendix. We use

$$\delta_p(y) \equiv w_0(y)/\tilde{w}(y) - 1 \tag{13}$$

to measure the *change in an individual agent's welfare* (expected lifetime utility) following the shock, where y is the agent's pre-shock portfolio; if $w_0(y) = w_0(y')$ whenever the two portfolios y and y' contain the same amount of silver, we use

$$\delta(z(y)) = \delta_p(y) \tag{14}$$

[7]Under the baseline parameters, the measure is $4 \times 10^{-12}\%$ at the steady state.

	m_k	0.25	0.5	1	2	4	6 / 8	12	60	Total
Baseline	Stock			1.000	1.182	3.942		28.80	0.075	35
	Circ.			0.486	0.033	0.007		$4e^{-12}$		0.527
	Mint.			0.252	0.395	0.453		0.310		1.409
Adding halfpenny	Stock		0.419	0.936	1.019	3.924		28.70	0.001	35
	Circ.		0.059	0.442	0.038	$1e^{-4}$		$4e^{-5}$		0.540
	Mint.		0.108	0.320	0.328	0.450		0.373		1.579
Adding halfpenny & farthing	Stock	0.250	0.357	0.905	0.991	4.104		28.39	0.001	35
	Circ.	0.061	0.054	0.433	0.043	0.007		$4e^{-4}$		0.599
	Mint.	0.065	0.126	0.323	0.342	0.432		0.318		1.607
Adding sixpence	Stock			1.000	0.526	2.624	1.973	28.80	0.075	35
	Circ.			0.491	0.025	0.009	0.009	$4e^{-12}$		0.536
	Mint.			0.253	0.262	0.708	0.689	0.310		2.221
Adding eightpence	Stock			1.000	1.182	0.017	7.827	24.90	0.075	35
	Circ.			0.489	0.031	0.002	$6e^{-6}$	$2e^{-11}$		0.522
	Mint.			0.252	0.395	0.010	0.974	1.063		2.694

Table 1: Pre-shock and post-shock steady states.

to measure the individual welfare change, where $z(y)$ is the amount of silver in the portfolio y. If the shock is a structure shock, these statistics measure the inconvenience for an individual agent when the coins added by shock are in complete shortage. If the shock is a debasement shock, these statistics measure the improvement for an individual agent due to the debasement. For comparison, we use

$$\Delta \equiv \pi \cdot w / \tilde{\pi} \cdot \tilde{w} - 1 \tag{15}$$

to measure the *change in the aggregate welfare*.[8] To emphasize, $(m_1, m_2, m_3, m_4) = (12, 4, 2, 1)$ is the pre-shock coinage structure and $F = 24$ in an exercise below unless indicated otherwise.

Structure shocks

Here we organize our results around four structure shocks, the *halfpenny, halfpenny-farthing, eightpence,* and *sixpence shocks* that add the halfpenny, halfpenny and farthing, eightpence, and sixpence, respectively, to the coinage structure. The eightpence, sixpence, halfpenny, and farthing are coins with 8, 6, 0.5, and 0.25 units of silver, respectively.

Table 1 provides an overview of the stocks, circulation volumes, and minting volumes of coins measured in silver units of the pre-shock steady state and the four

[8] An alternative aggregate statistic is $\tilde{\pi} \cdot w_0 / \tilde{\pi} \cdot \tilde{w} - 1$. In all our exercises, the two aggregate statistics are in the same order of magnitude. We focus on $\pi \cdot w / \tilde{\pi} \cdot \tilde{w} - 1$ because $\tilde{\pi} \cdot \tilde{w}$ ($\pi \cdot w$, resp.) is the ex-ante welfare for each agent in the pre-shock (post-shock, resp.) economy when he draws his initial portfolio from the distribution $\tilde{\pi}$ (π, resp.).

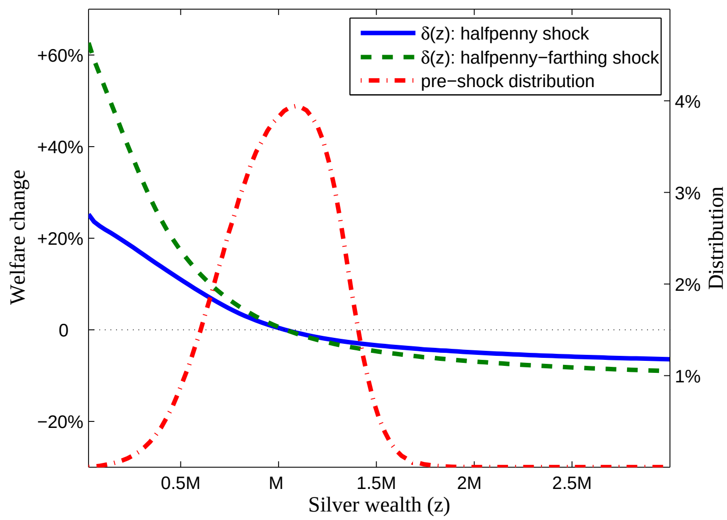

Figure 1: Left axis: changes in individual welfare ($\delta(z)$) under the halfpenny structure shock and the halfpenny-farthing structure shock. Right axis: steady-state distribution before the shocks.

post-shock steady states. A couple of remarks are in order. First, shillings and jewelry together absorb more than 70% of silver and almost all silver in this proportion is not used for the transactional purpose. This proportion does not vary much when we vary m_0 from 60 down to 30 but the split between shillings and jewelry may vary substantially. Second, while coins larger than pennies facilitate less than 3% of the total transaction values, they contribute to more than 70% of the total minting volume; that is, a larger minting volume need not imply that the corresponding coin is more useful in transactions.

The key statistic for each shock is the change in the individual agent's welfare $\delta(z)$ defined by (14). For the eightpence and sixpence shocks, these statistics are positive for all individuals but bounded above by 0.001%. The lifetime improvement of an agent who benefits the most from these shocks is offset by the costs to carrying 70 coins into the decentralized market once. So filling in the gap between the shilling and groat benefits everyone but no one would be bothered much if the gap is left there. For each of these two shocks, the change in the aggregate welfare Δ defined by (15) is around 0.001%, which is largely indicative of inconvenience felt by an individual agent when the coins in concern are in shortage.

Figure 1 displays the two $\delta(z)$ curves for the halfpenny and halfpenny-farthing shocks. The two curves share the very same patterns. The change in an agent's welfare is decreasing in his pre-shock wealth, agents in the poor side get great improvements, and agents in the rich side are worse off. For the halfpenny shock, $\Delta = 1.43\%$ and $\delta(z)$ ranges from 25.21% to -6.43%. For the halfpenny-farthing shock, $\Delta = 1.71\%$ and

$\delta(z)$ ranges from 62.66% to −9.02%. For these two shocks, the aggregate statistics highly underestimate inconvenience felt by poor people when coins in concern are in shortage. The difference between the two $\delta(z)$ curves in Figure 1 gives a measurement of the marginal effect from adding the farthing when the halfpenny is available. For an alternative measurement, we study the *alternative farthing shock* that adds the farthing to the coinage structure $(m_1, m_2, m_3, m_4, m_5) = (12, 4, 2, 1, 0.5)$. The aggregate statistics is $\Delta = 0.27\%$ and the $\delta(z)$ curve is very close the difference between the two $\delta(z)$ curves in Figure 1.

The patterns of $\delta(z)$ in Figure 1 may be explained by the consumption-smoothing effect and a countering effect. To see the former effect, suppose an agent spends one unit of the smallest coin when he is a buyer.[9] If his present wealth is z, then his lifetime utility can be written as

$$\sum_{t=1}^{z/m_*} (\frac{0.5}{1-0.5\beta})^t \beta^{t-1} u(c_t), \tag{16}$$

where c_t is his consumption when his wealth is $z-(t-1)m_*$ ($m_* = \min_{k\geq 1} m_k$). One may interpret (16) as that the agent spreads his purchasing power over z/m_* periods. Suppose a shock does not affect the agent's purchasing power. But with a reduction in m_*, the agent benefits because he can spread his consumption over more periods. Because of discounting, a smaller z means a larger consumption-smoothing benefit. To understand the countering effect, note that the amount of goods received by a buyer is decreasing in his partner's reservation value when the buyer transfers the same amount of silver in the payment. Because consumption smoothing benefits all agents, it tends to raise that reservation value, which, in turn, reduces the buyer's surplus from trade. The countering effect may be the dominant one for rich agents as it may not vary much across agents.

The tale of two sides (due to shortages of small coins) in Figure 1 is of great interest. But when agents meet more frequently, i.e., F and β become larger, the consumption-smoothing effect may be strengthened to dominate the countering effect for people in the rich side. A larger F works through two channels. First, it weakens the influence of discounting. Secondly, as is shown in Table 1, when F is at the baseline value, pennies play the dominant role in transactions and, adding coins smaller than pennies has an observable but not dramatic effect on the usage of pennies. So the consumption pattern in (16) only applies to agents in the poor side after the shock. The larger F increases the measure of agents who spend one unit of the smallest coin in the decentralized market after the shock. In other words, the larger F leads to a larger proportion of agents who take full advantage of the spread of consumption permitted by the addition of coins smaller than pennies.

For a shock to induce a positive $\delta(z)$ curve, F needs to exceed some level that depends on the pre-shock m_*. When $F = 48$, the halfpenny shock yields $\Delta = 28.75\%$

[9] This may happen in equilibrium if β is sufficiently close to unity when $\gamma = 0$ and $m = (m_0, m_1) = (\infty, 1)$; see Camera and Corbae [2].

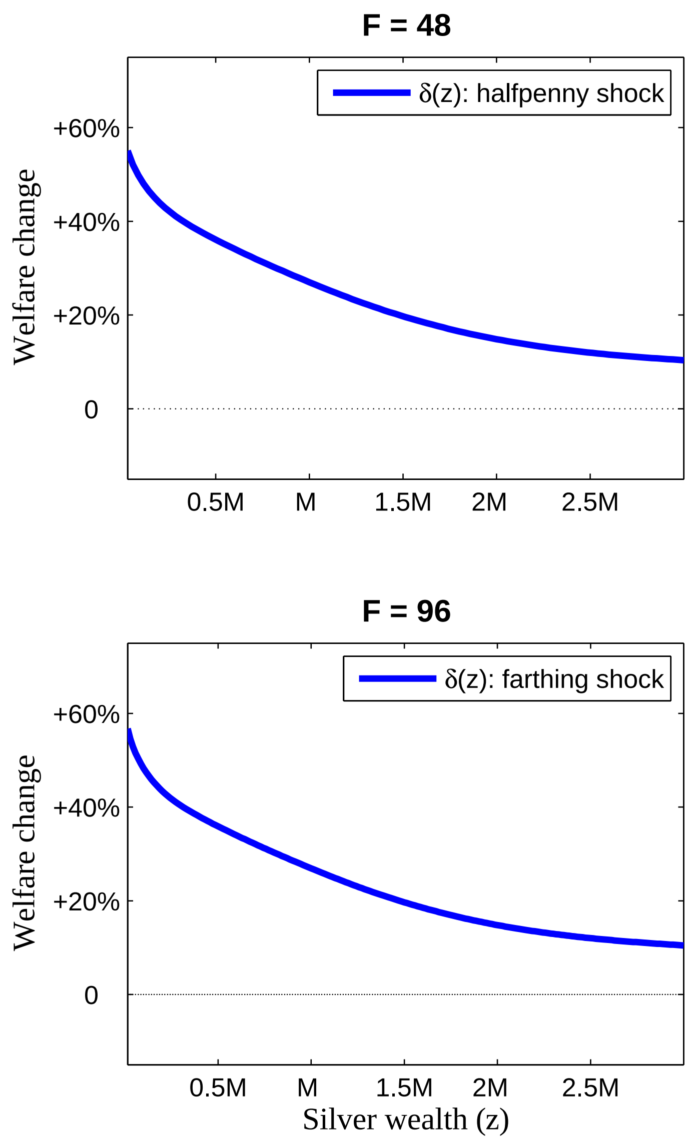

Figure 2: Changes in individual welfare ($\delta(z)$) under the halfpenny structure shock with $F = 48$ (upper); and the alternative farthing structure shock with $F = 96$ (bottom).

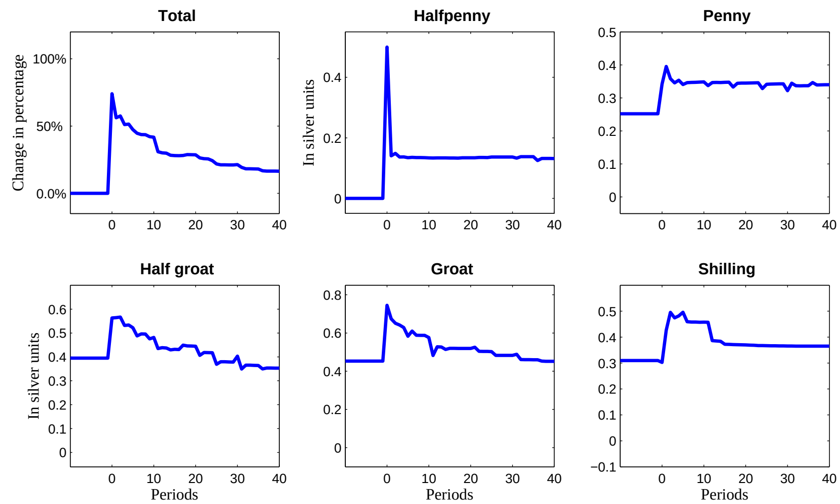

Figure 3: Minting volume responses following the halfpenny structure shock.

and the $\delta(z)$ curve in the upper row of Figure 2 that ranges from 55.06% to 10.37%; when $F = 96$, the alternative farthing shock yields $\Delta = 28.60\%$ and the $\delta(z)$ curve in the bottom row of Figure 2 that ranges from 56.76% to 10.48%. The new tale points to a universal unhappiness. In fact, the universal unhappiness prevails even when the farthing is available, as long as the trade is sufficiently frequent. To make the point, we study the *alternative halffarthing shock* that adds $m_7 = 0.125$ to the coinage structure $(m_1, m_2, m_3, m_4, m_5, m_6) = (12, 4, 2, 1, 0.5, 0.25)$. When $F = 240$, $\delta(z)$ ranges from 66.65% to 19.97% and $\Delta = 39.05\%$.

Structure shocks as special debasement shocks

A structure shock may be viewed as a special debasement shock; for example, the halfpenny shock is equivalent to a shock that debases the penny by 50% while mints the coin with 1 gram of silver as the zenny. From this perspective, we relate the minting and usage of coins in the post-shock equilibrium to the debasement puzzle. First, coins in the pre-shock and post-shock coinage structures cocirculate by weight following each shock and, one can see from Table 1 cocirculation even persists in the long run. Secondly, each shock induces large increases in the minting volume in the post-shock equilibrium (compared to the pre-shock steady state). The increases are presented in Figures 3 and 4 for the halfpenny and sixpence shocks, respectively. Aside from some details, the patterns in Figure 3 apply to the halfpenny-farthing shock and the patterns in Figure 4 apply to the eight-pence shock.

When halfpennies are added, the minting volume in the post-shock steady state increases by 12% and halfpennies contribute to more than 80% of that increase. The transition to the post-shock steady state is gradual. In each of the first 10 periods,

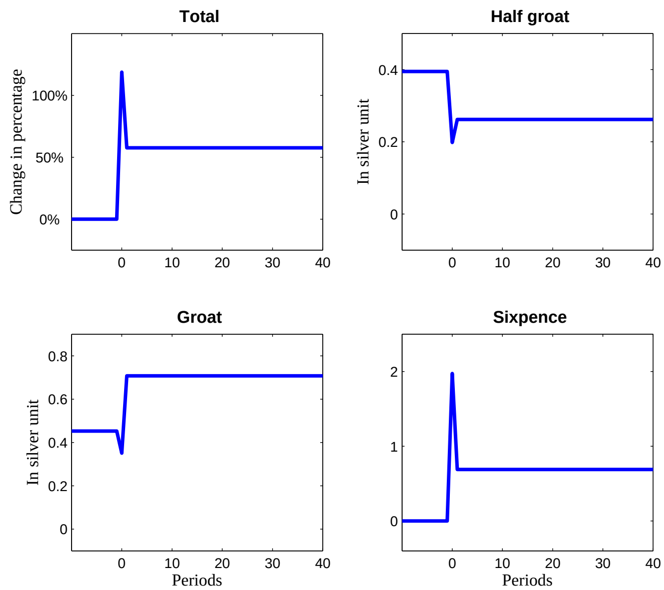

Figure 4: Minting volume responses following the sixpence structure shock.

the minting volume increases by more than 40% and all coins contribute substantially. This transitional process may be explained as follows. In equilibrium, a buyer holds one or two halfpennies for the transactional purpose but a seller holds a halfpenny only when his silver wealth is not an integer. When a buyer melts jewelry or other coins in exchange for halfpennies at period 0, he tends to have extra silver which can only be used to mint other coins. Because many agents need to adjust holdings of halfpennies after period 0, the extra minting of other coins due to the extra silver from minting halfpennies lasts for multiple periods.

When sixpences are added, the minting volume in the post-shock steady state increases by 57% and sixpences contribute to more than 80% of that increase. The transition to the post-shock steady state is almost instantaneous for two reasons. First, an agent can support the period-0 minting of sixpences by half groats, groats, and other coins that are used to mint half groats and groats in the pre-shock steady state; that is, his minting of sixpences at period 0 may only affect the minting of half groats and groats. Second, the agent's choice of sixpences, groats, and half groats at the mint is not sensitive to his type (because those coins are not frequently used in transactions); that is, there is little need for him to adjust holdings of these coins at period 1.

	m_k	0.5	1	2	4	6 / 8	12	60	Total
	Stock		1.000	1.182	3.942		28.80	0.075	35
Baseline	Circ.		0.486	0.033	0.007		$4e^{-12}$		0.527
	Mint.		0.252	0.395	0.453		0.310		1.409
Debasing	Stock	1.317		1.056	3.924		28.70	0.001	35
penny	Circ.	0.485		0.066	0.034		$1e^{-4}$		0.585
(by 50%)	Mint.	0.298		0.367	0.457		0.374		1.496
Debasing	Stock		1.000	0.204	3.911	29.81		0.075	35
shilling	Circ.		0.486	0.021	0.026	0.039			0.572
(by 50%)	Mint.		0.252	0.203	0.974	1.085			2.514
Debasing	Stock		1.000	1.182	2.038	30.70		0.075	35
shilling	Circ.		0.487	0.033	0.005	$6e^{-6}$			0.525
(by 33%)	Mint.		0.252	0.395	0.351	0.376			1.374

Table 2: Steady states before and after the debasement shocks.

Debasement shocks

Compared with a structure shock that is equivalent to a special debasement shock, a debasement shock has the feature that in the post-shock economy the mint does not supply coins with the same silver content as some old coins. As it turns out, this feature imposes a problem for our computation. That is, the values of old coins may be highly sensitive to the measures of old coins in circulation and, as a result, our algorithm may fail to converge. To deal with the problem, we need to choose a not very large $\hat{t}$ for old coins to exit. We present our results with $\hat{t} = 50$.

Our main interest is whether the indicated feature of a debasement shock may substantially alter the welfare effects and the post-shock minting and usage of coins that are observed from a corresponding structure shock. To this end, we study the *penny debasement shock* that debases the penny in the coinage structure by 50%, and the *shilling debasement shocks* that debase the shilling by 50% and 33%.

Table 2 summarizes the stocks, circulation volumes, and minting volumes of coins measured in silver units of the pre-shock steady state and the three post-shock steady states. A notable feature is that following the penny debasement, much of silver occupied by jewelry is released to coins even though a new penny contains a less amount of silver than an old penny (following each shilling debasement only a tiny amount of silver occupied by jewelry is released). In other words, the penny debasement makes holding silver in money more attractable than holding silver in jewelry.

Regarding the post-shock equilibrium outcomes, the penny debasement shock resembles the halfpenny structure shock and the two shilling debasement shocks resemble the sixpence and eightpence structure shocks in the welfare effects and minting activities. For all the debasement shocks, $\delta_p(y) \approx \delta_p(y')$ (see (13)) when $z(y) = z(y')$. When the penny is debased, $\Delta = 1.43\%$ and $\delta_p(y)$ ranges from 25.21% to -6.44%.

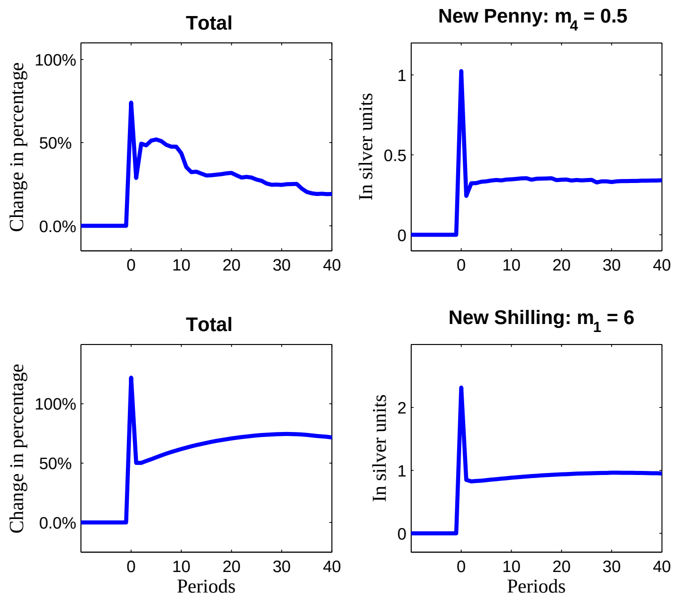

Figure 5: Minting volume responses following the debasement shocks. Upper row: debasing the penny from 1 to 0.5; bottom row: debasing the shilling with from 12 to 6.

When the shilling is debased, the values of $\delta_p(y)$ and Δ are all negative, with $\delta_p(y)$ bounded below by -0.008% and Δ by -0.007%; the negative (but insignificant) effects may be attributed to the fact that old shillings are a more convenient store of value than new shillings. Figure 5 presents increases in the minting volume following the penny debasement and following the 50% shilling debasement.

Following each debasement shock, we observe cocirculation of old and new coins before old coins exit. An interesting finding pertains to the difference between circulation of old shillings and circulation of old pennies. After the shilling is debased, old shillings get more and more circulated because people can only get this convenient store of value from the decentralized-market trade. After the penny is debased, old pennies get less and less circulated because new pennies are good substitutes and more and more old pennies are melted in exchange for new pennies. Figure 6 presents the different patterns when the penny is debased and when the shilling is debased by 50%.

4 Two extensions

Here we study two extensions of the basic model. In one extension, the small-change problem may be a considerable part of the small-coin problem. In another extension, the small-coin problem persists while small coins do not dominate in transactions.

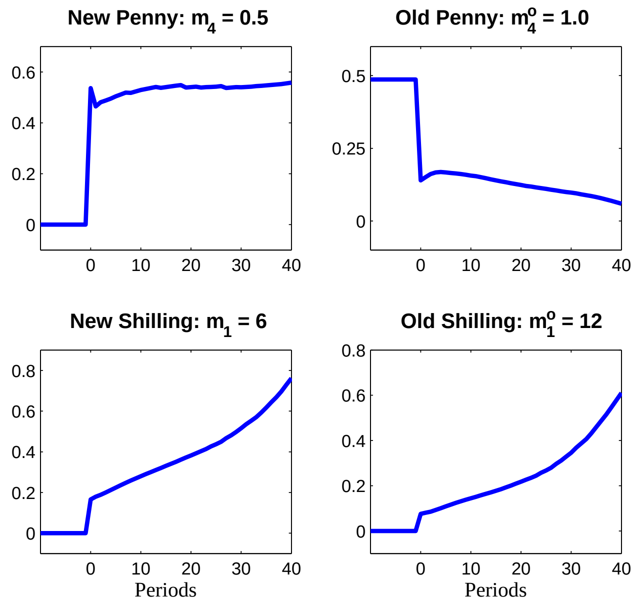

Figure 6: Circulations of coins following the debasement shocks. Upper row: debasing the penny from 1 to 0.5; bottom row: debasing the shilling from 12 to 6.

Small change and wasted trading opportunities

The folk theory in introduction emphasizes the small-change role of small coins. In medieval documents, we may also see the complaints that people had to waste some trading opportunities because of the small-change problem. An often-cited petition to the England king in 1444 asserted that

> people, which would buy such victuals and other small things necessary, may not buy them, for default of half pennies and farthings not had on the part of the buyer nor on the part of the seller. (Ruding [16, p. 275])

We may appeal to a structure shock that adds a small coin to measure wasted trading opportunities as follows. If two agents in a meeting do not trade in the pre-shock steady state but they trade at period 0 in the post-shock equilibrium, then we say that two agents waste the trading opportunity in the pre-shock steady state. We use the mass of such meetings to measure wasted trading opportunities because of the small-change problem. In the basic model, the measure of wasted trading opportunities is pretty low. For example, it is 0.04% when we apply the halfpenny shock at the baseline F. This means that for a fixed pair of agents, if the buyer does not spend a penny in the pre-shock steady state, he tends to have no sufficient incentive to spend a halfpenny in the post-shock equilibrium. In other words, no trade in the pre-shock meeting is not so much because a penny is too big.

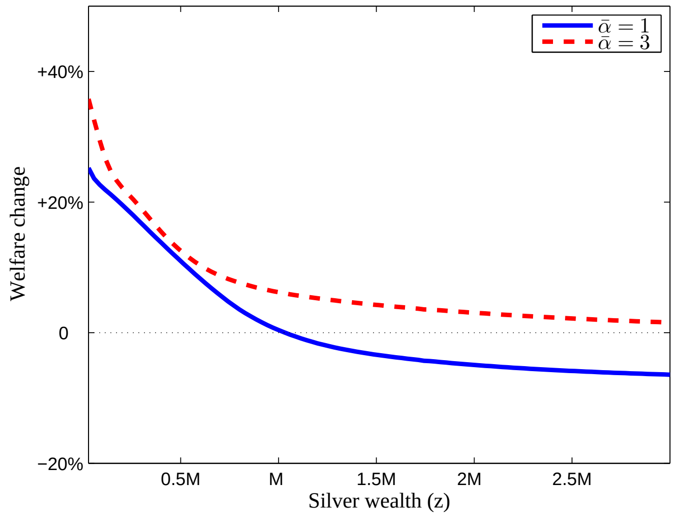

Figure 7: Changes in individual welfare($\delta(z)$) under the halfpenny structure shock with idiosyncratic preference shocks. $\bar{\alpha} \in \{1, 3\}$.

To capture the small-change role of small coins, we study a simple extension of the basic model that may have sufficient wasted trading opportunities. Suppose each buyer receives an i.i.d. idiosyncratic preference shock in his pairwise meeting. His utility from consuming x is $\alpha u(x)$ given the realization of the shock is α, and the shock is distributed over $\{1, ..., \bar{\alpha}\}$. For quantitative exercise, we consider the uniformly-distributed preference shock and experiment with $\bar{\alpha} = 3$. Applying the halfpenny shock, we find that the wasted trading opportunities due to the shortage of halfpennies are 24%. Figure 7 displays the $\delta(z)$ curve for the halfpenny shock (the solid line). Compared with the basic model ($\bar{\alpha} = 1$), the addition of the halfpenny has a much strengthened effect.

To get a better sense of the above experiment, consider a pairwise meeting in the pre-shock steady state that agents trade when α is large but does not when α is small. The mass of those meetings is about 25%. In such a meeting, the buyer skips the present trading opportunity, anticipating the higher future payoffs when α becomes large. But when the halfpenny is added, the buyer tends to have a sufficient incentive to spend the halfpenny under the small α, meaning that the halfpenny plays a small-change role.

Higher nominal GDP and more usage of large coins

For the part of history in concern, the annual nominal GDP per capita in England fell in the range from 200 to 400 pence. Likely, in a medieval economy, a substantial portion of GDP was not realized through the market transactions and, there was some intrinsic heterogeneity that permitted a small class of people to procure a large proportion of GDP. Our model may be better interpreted as the part of economy that excluded that small class of people and excluded the non-market transactions. With this interpretation, one may target the annual nominal GDP at 100 pence.[10]

The basic model yields the annual nominal GDP at 12 pence with the baseline parameters. But it can match any pre-set nominal GDP level. Indeed, if we double the meeting frequency F, we double the nominal GDP. Moreover, as noted above, when F rises, there is a large welfare gain if a shock introduces coins smaller than the extant smallest coins. But when F rises, agents also have a strong tendency to use the smallest coins. Would the nominal GDP not come at the expense that all coins greater than the smallest coins are out of circulation? This motivates us to study the following extension.

Suppose each meeting at period t consists of $N+1$ phases. The first phase is phase 0 and the last is phase N. If two agents stay together at the start of a phase n, then each agent can choose to depart at the phase and, moreover, they may be forced to depart at the end of the phase by an i.i.d departing shock that is realized at the start of the phase; the departure probability implied by the shock is $1-\rho_n$. We let $\rho_0 = 1$, $\rho_n < 1$ for $N > n \geq 1$, and $\rho_N = 0$. If either agent departs, both agents are idle in the rest of period t. If two agents stay together at phase $n \in \{1, .., N\}$, the seller can produce a good that is consumed by the buyer at the phase. The seller cannot produce at phase 0. The buyer's utility from consuming the bundle $(x_1, ..., x_n)$, $n \leq N$, is $\sum_{i=1}^{n} u(c_i)$; the seller's disutility from producing the bundle is $\sum_{i=1}^{n} c_i$. When a period is short, say, it is just one day, then we may think that the N goods are physically distinct (a buyer buys bread, butter, and milk from a seller in the period). When a period is more than N days, we may think that the N goods in the meeting as N time-indexed goods (the buyer buys bread N times from the seller during the period).[11]

The buyer makes a take-it-or-leave-it offer at phase 0. The offer is a contingent plan in that if agents are forced to depart by the end of phase n, the buyer is to make a payment $\iota_n \in L(y_b, y_s)$ (see (8)) for the consumption bundle $(x_1, ..., x_n)$. The agents are committed to the plan in that if they stay together, then the seller is to deliver the good and the buyer is to deliver the payment as in the plan. The commitment is limited in that each agent can choose to depart anytime. In terms of w_{t+1}, the trade in a pairwise meeting between a buyer with y_b and a seller with y_s solves the optimization problem

$$\tilde{f}(y_b, y_s, w_{t+1}) = \max_{(c_1, ..., c_N, \iota_1, ..., \iota_N)} \sum_{n=1}^{N} \mu_n \left[\sum_{i=1}^{n} u(c_i) + \beta w_{t+1}(y_b - \iota_n) \right]$$

[10]If a household has 5 members, this means that the household annually receives monetary incomes around 500 pence, which may be close to the historical data.

[11]We may follow Shi [21] to assume that the buyer needs a consumption device to consume and he surrenders the device to the seller as a collateral if he is to come back next time.

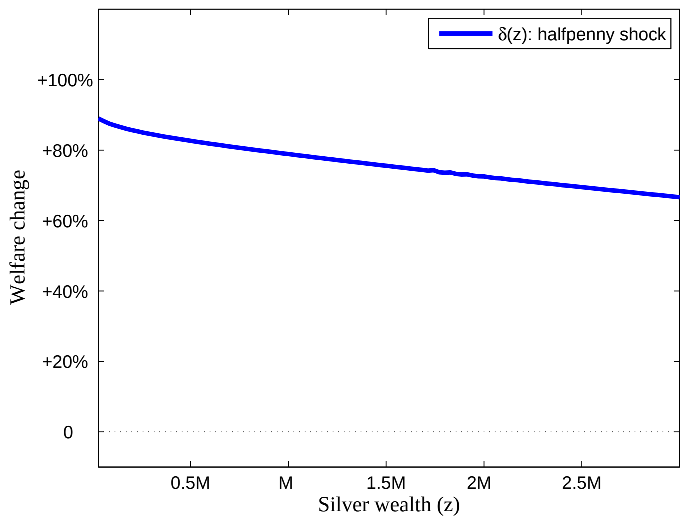

Figure 8: Changes in individual welfare ($\delta(z)$) under the halfpenny structure shock, with $F = 72$, $N = 3$ and $\rho_1 = \rho_2 = 0.9$.

subject to $\iota_n \in L(y_b, y_s)$, $1 \leq n \leq N$, and

$$-c_n + \beta w_{t+1}(y_s + \iota_n) \geq \beta w_{t+1}(y_s + \iota_{n-1})$$

where $\iota_0 = 0$, $\mu_n = (1 - \rho_n) \prod_{i=0}^{n-1} \rho_i$ for $1 \leq n \leq N - 1$, and $\mu_N = \prod_{i=0}^{N-1} \rho_i$.[12]

In equilibrium, the buyer's utility and payments in a meeting are roughly proportional to the number of phases when the buyer and seller stay together.[13] This leaves a room to manipulate N and $\{\rho_n\}$ to get the desired distribution of the circulation volumes of coins before a structure shock and, in the meanwhile, does not weaken the effect of consumption smoothing when the shock adds coins smaller than the existing smallest coins. In other words, agents may largely get around of small coins in transactions but there is still a small-coin problem. For example, we want pennies, groats and half groats are more or less equally used in transactions under the baseline coinage structure. We experiment with $F = 72$, $N = 3$, and $\rho_1 = \rho_2 = 0.9$; one interpretation is that people meet once every 5 days and 10% of meetings lasts for 1 day, 9% for 2.5 days, and 81% for 5 days. The steady-state nominal GDP is around 96 pence per year; the circulation volumes of the penny, half groat and groat, respectively, are 0.431, 0.599 and 0.481 units of silver in each period of trade; and the shilling is purely a store of value. Figure 8 displays $\delta(z)$ under the halfpenny shock.

[12]Given w_{t+1}, $\{\rho_n\}$ affects the buyer's objective function but does not affect the seller's participation constraints in the optimization problem. If the departing shock of a phase is realized after the seller produces but before the phase ends, then $\{\rho_n\}$ also affects the seller's participation constraints.

[13]If the buyer and seller know the number of staying-together phases at the start of the meeting and the number is determined by a random variable, then given a large F, the buyer tends to skip the trade when the number is small and to spend the minimal amount when the number is large.

5 Discussion

Here we first discuss our model and results. We follow Lee, Wallace, and Zhu [9] closely in setting up our basic model. Attractiveness of this modelling choice is that the Lee-Wallace-Zhu model itself is built on a model not designed for the historical coinage issues. Indeed, if we set $\gamma = 0$ (zero carrying costs) and $m = (m_0, m_1) = (\infty, 1)$ (fiat money with one denomination), then the basic model turns into the version of the familiar model of Trejos and Wright [22] and Shi [20] that is studied by Zhu [28]. With some minimal departure from the plain version of the Trejos-Shi-Wright model, our model delivers a rich set of implications for the coinage issues in concern.

The most striking implications are the individual welfare losses due to shortages of small coins. In the context of our model, a critical parameter is the silver stock M. Our choice of M is explained above. A local change in M, say, from 35 to 40, does not affect the relevant numbers much. We have not found a way that can efficiently redo all the above exercises for a large change in M, say, from 35 to 100; the computational burden increases dramatically in some exercises. To give some idea of what may happen for a larger M, we note that the alternative farthing shock above is almost identical to the structure shock that adds the halfpenny to the coinage structure $(m_1, m_2, m_3, m_4, m_5) = (24, 8, 4, 2, 1)$ when $M = 70$, and the alternative halffarthing shock above is almost identical to the structure shock that adds the halfpenny to the coinage structure $(m_1, m_2, m_3, m_4, m_5, m_6) = (48, 16, 8, 4, 2, 1)$ when $M = 140$.[14] Our model, however, may exaggerate the welfare losses because a medieval person's consumption did not all come from monetary transactions. Suppose monetary transactions only contributed to one third of the consumption. Suppose the consumption of goods and services from monetary transactions entered into the person's utility function as an object distinct from the consumption of goods and services from other means (e.g., barter, credits, and self production). Then, one may discount a welfare number by 2/3 to get a more realistic estimation, which may remain significant. The significance may well explain the historical experimentation with different sorts of imperfect substitutes to full-bodied small coins.

Perceivably, the basic model and the two extensions in section 4 cannot capture all relevant monetary aspects in a late-medieval European economy. For example, medieval mints charged people to cover the labor and material costs and collect seigniorage. The minting fees would contribute to shortages of small coins. For, given the minting technology, it was much more costly to produce farthings than shillings. But given the minting fees permitted by kings, mints might not produce farthings as demanded because it would be much more profitable to produce shillings than farthings (see Redish [12, p. 113]). Also, bimetallism was typical in late medieval Europe: gold was largely used in high-value transactions and silver was mostly used in

[14] Also, for $m = (m_0, m_1) = (\infty, 1)$ and $\gamma = 0$, we work on the shock that reduces m_1 from 1 to 0.5; the $\delta(z)$ curves for $F = 24$ and $F = 120$ are similar to those in Figures 1 and 2, respectively.

the daily life. The carrying cost of monetary objects may play a more significant role in bimetallism. For, the trade facilitated by gold may have a much higher nominal value in silver units. To address the influence of the minting fees, we may assume that each agent needs to incur some amount of disutility to obtain a unit of coin. We may further assume that the mint sets an upper bound on the aggregate minting volume for each type of coin. This bound may be exogenous but it can be endogenous as the mint's optimal choice given the minting fees, the mint's resource, and the minting demand. Either way, a binding bound may describe partial shortages of small coins in history. This extension, however, is much more challenging to analyze quantitatively because the state space increases dramatically. To study bimetallism, we may follow Wallace and Zhou [27] by assuming that there are two types of agents who permanently differ in productivity as sellers. Intuitively, agents with high productivity may tend to use gold coins. But the equilibrium outcomes in this extension may much depend on the details of how two types of people interact. Both extensions are left for the future work.

Next we turn to the related literature. In the economic literature, shortages of coins or small coins are dealt with by Sargent and Velde [18] and a few other papers. Wallace and Zhou [27] study a matching model in which there is a unit upper bound on money holdings, some agents are less productive than other agents, and all coins are kept by the set of more productive agents in a steady state; they identify a shortage of coins with the concentration of wealth. Kim and Lee [6] compare the steady-state aggregate welfare in the matching model studied by Zhu [28] with the steady-state aggregate welfare in a commodity money version of that model; they identify a shortage of small commodity-money coins with a part of the welfare difference contributed by that commodity-money coins are more valuable than fiat-money coins.

Lee and Wallace [8] compare the steady-state aggregate welfare in the matching model studied by Zhu [28] by varying the size of the penny. They include the cost of maintaining monetary objects in their analysis. They conclude that medieval Europe might set the size of the penny right. We suspect that poor agents in their model should get great improvements if the size of the penny is reduced. Redish and Weber [13] build a bimetallic system into the model of Lee, Wallace, and Zhu [9]. Focusing on steady-state comparison, Redish and Weber [13] identify a shortage of small coins with the improvement in the steady-state aggregate welfare when a smaller coin is added. While their model is similar to ours, their parameterization is quite different. In their exercises, the average of coin holdings is no more than 10 but the consumption-smoothing effect from an addition of small coins does not stand out. We suspect that this is mainly because they set $F = 1$.

There is a small economic literature that tackles the debasement puzzle. In a cash-in-advance model, Sargent and Smith [17] assume that new and old coins circulate by tale. Under this assumption, agents bring all old coins into the mint in exchange for new coins. On the empirical ground, Rolnick, Velde, and Weber [15] argue that by-tale circulation violates facts documented in the debasement puzzle and that by-tale circulation would have induced a much larger minting volume than observed (the data indicates that only a portion of old coins were recoined). In matching models with one unit upper bound on coin holdings, Velde, Weber, and Wright [25] and Li [10] use side payments offered by the mint as incentives for people to bring in old coins in exchange for new coins at a one-to-one rate. None of these three models is suitable to study the demand for multiple monetary objects.

All three models concern Gresham's law. Partly rooted in medieval debasements, Gresham's law says that bad money (new coins) drives out good money (old coins).[15] The renowned law has long been known to be ambiguous at best. On the theoretical ground, it relies on the circulation-by-tale assumption that is effectively imposed from the outset; on the empirical ground, there are numerous counterexamples (see Rolnick and Weber [14] and Velde [23]). In each of the three models, Gresham's law is not universal but good money is driven out by bad money at some parameter space because of asymmetric information (Velde, Weber, and Wright [25]), the government transaction policy (Li [10]), or the circulation-by-tale assumption (Sargent and Smith [17]). In our model, some old coins are melt (i.e., some good money is driven out) but other are kept (up to the exit period $\hat{t}$) following each debasement shock we study.

6 Concluding remarks

Commodity money occupies the most part of monetary history in civil societies. Compared with fiat money, commodity money is primitive in that its service as money seems much constrained by its physical properties such as scarcity, portability, divisibility, and recognizability. Although it is a conventional wisdom that these properties matter, not much has been explored probably because it is not easy to place them in models that many economists are used to. In other words, it is the primitiveness of commodity money that presents a challenge to modern monetary economics. Here we explore some implications of the primitiveness in an off-the-shelf model. A general message is that the small-coin problem is costly but the problem would be solvable if a tiny amount of precious metal were portable and recognizable. While no such metal exists, there may exist such a "commodity" (e.g., bitcoins).

[15]Fetter [5] describes how Gresham's law was reformed in the nineteenth century from a comment on debasements made by Gresham in 1558.

Appendix

A Proof of Proposition 1

The proof applies the standard fixed point argument. For existence of an equilibrium for a given π_0, it is routine to (i) construct a set S that is compact in the product topology and an element of which is a sequence $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$, (ii) construct a mapping F from S to S that is implied by the definition of equilibrium and whose fixed points are equilibria, and (iii) verify that all conditions for the application of Fan's fixed-point theorem are satisfied. So there exists an equilibrium. To show that this equilibrium is a monetary equilibrium, suppose by contradiction the opposite. Without loss of generality, suppose that some agent holds silver wealth B at date 0 and all his wealth is in jewelry. Consider two options of this agent when he is a buyer: minting one unit of the smallest coin and no minting any coin. For the first option, his expected payoff is bounded below by

$$-\gamma_C + (1 - \frac{0.5M}{B - m_*})u[\frac{\beta\left(v(B) - v(B - m_*)\right)}{1-\beta}] + \frac{\beta}{1-\beta}v(B - m_*).$$

Notice that $1 - 0.5M/(B - m_*)$ is a lower bound on the measure of sellers whose wealth levels in silver do not exceed $B - m_*$ and the agent can receive at least $\beta\left(v(B) - v(B - m_*)\right)/(1-\beta)$ amount of the good from such a seller. For the second option, his expected payoff is $v(B)/(1-\beta)$. But then (12) implies the first option has a higher payoff, a contradiction. Existence of a monetary steady state can be proof by essentially the same argument.

B Numerical algorithms

B.1 Computing a steady state

To begin with, vectorize the $K+1$-state space into a one-dimensional state, and define the value vectors $\{w, g\}$ and distribution vectors $\{\theta, \pi\}$, $\theta = (\theta^b, \theta^s)$, accordingly. Denote the total possible number of states as S.

1. Begin with an initial guess $\{w^0, h^0, \theta^0, \pi^0\}$, where π^0 and θ^0 are consistent with the total silver stock M.

2. Given end-of-stage-1 value h^i and beginning-of-stage-1 distribution π^i from i-th iteration, solve the problem (4), and use the solution to update beginning-of-stage-1 value w^{i+1} and end-of-stage-1 distribution θ^{i+1} .

3. With w^{i+1} and θ^{i+1}, solve the problem as described in (5). Record the terms of trade of each relevant pairs, and update h^{i+1} and π^{i+1} accordingly.

4. Repeat step 2-3 until the convergence criterion is satisfied: $\|w^{i+1} - w^i\| < 10^{-6}, \|h^{i+1} - h^i\| < 10^{-6}$ and $\|\theta^{i+1} - \theta^i\| < 10^{-8}, \|\pi^{i+1} - \pi^i\| < 10^{-8}$.

B.2 Computing a post-shock equilibrium

The computation for the transition path is essentially about iterations on the series of $\Psi \equiv \{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=1}^T$, $h_t = (h_t^b, h_t^s)$ and $\theta_t = (\theta_t^b, \theta_t^s)$, where T is the number of periods it takes for the economy to reach a new steady state. Before computing the transition paths, we first need to compute the post-shock steady state using an algorithm similar to B.1, with the change that choice of portfolios containing old coins are eliminated at the minting stage. Denote this steady state as $\{w_T, h_T, \theta_T, \pi_{T+1}\}$. We also have to translate the distribution from the pre-shock steady state, into the beginning distribution in the debasement environment, denote the beginning distribution as π_1.

1. Take an initial guess $\Psi^0 \equiv \left\{w_t^0, h_t^0, \theta_t^0, \pi_{t+1}^0\right\}_{t=1}^T$, with $w_T^0 = w_T$.
2. Start from the last period T. Given w_T and θ_T^i, solve the pairwise bargaining problem as described in (5), and get h_T^i. Record the implied Markov transition matrix as Λ_T^i. Use h_T^i and π_T^i, solve the problem of minting, and get w_{T-1}^i accordingly. Record the implied Markov transition matrix as Υ_T^i. Then use w_{T-1}^i and θ_{T-1}^i, repeat the previous procedure for problems in period $T-1$. Finally, we will have a new series $\{w_t^i, h_t^i\}_{t=1}^T$. And then use $\{\Lambda_t^i, \Upsilon_t^i\}_{t=1}^T$ and π_1 and generate a new series of distributions $\left\{\pi_t^{i+1}, \theta_t^{i+1}\right\}_{t=1}^T$.
3. Now use $\left\{\pi_t^{i+1}, \theta_t^{i+1}\right\}_{t=1}^T$ and w_T, repeat Step 2 and get $\left\{\pi_t^{i+2}, \theta_t^{i+2}\right\}_{t=1}^T$.
4. Repeat 2-3 until the convergence criterion is met: $\max_t \left(\left\|\pi_t^{i+1} - \pi_t^i\right\|\right) < 10^{-8}$, $\max_t \left(\left\|\theta_t^{i+1} - \theta_t^i\right\|\right) < 10^{-8}$, $\max_t \left(\left\|w_t^{i+1} - w_t^i\right\|\right) < 10^{-6}$, and $\max_t \left(\left\|h_t^{i+1} - h_t^i\right\|\right) < 10^{-6}$.

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