The Risk-sharing Role of Multiple Outside Assets

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Abstract

We study the risk-sharing role of multiple outside assets in a two-sector Lagos-Wright economy with aggregate shocks. Using a mechanism-design approach, we show how a planner can implement the best allocation achievable with memory (perfect monitoring) by employing a state-contingent asset-supply policy and a trading protocol that endogenizes no asset-substitution. The optimal policy requires active management of relative asset prices under large shocks or impatience. If the two assets are the currencies of two countries, then the optimum is neither fixed nor floating exchange rates.

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Key Words: Nominal assets; State-contingent policy; Risk sharing; Optimal trading protocol; Exchange and interest rates

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1 Introduction

A central question in monetary economics is how arrangements involving multiple assets can improve welfare. In an international context, this includes exchange-rate policy; in a one-country context, this includes interest-rate policy. This question is often addressed in models where the distinct functions of the assets are imposed exogenously (see, e.g., Lucas [6] and Lucas and Stokey [7]). In this paper, the rationale for multiple assets arises from their essential role in facilitating risk sharing. Our work has two distinct features. First, we use a background model in which money has a role as a substitute for perfect monitoring or memory (see Townsend [11] and Kocherlakota [4]). Second, following Zhu and Wallace [17], we achieve no asset-substitution in a way that is consistent with having trade in each meeting between people in the model be in the meeting-specific core, a desirable feature that does not hold in models with country-specific or sector-specific cash-in-advance constraints and competitive trade. The specific model we use is a two-sector version of the Lagos-Wright [5] model with sector-specific aggregate shocks.[1]

The model is an infinite-horizon model with two stages at each discrete date. Stage-1 is a centralized meeting with a linear good; stage-2 has people meeting in pairs with general costs of production and utilities of consuming. In our version, there are two stage-2 sectors that are distinguished by sector-specific productivity and taste shocks in the pairwise meetings. The two outside assets are either two currencies (when each sector is a country) or currency and a one-period discount government bond (when the two sectors are part of a single country). In each case, a planner supplies assets in the centralized meeting of the model at a state-dependent relative price of the two assets—the nominal exchange rate in the first case, the nominal interest rate in the second case. The model lends itself to the study of an *ex ante* representative-agent notion of welfare. When there is sufficient impatience for given shocks or when the shocks are large enough for a given degree of impatience, the best policy has roles for both assets and is state-dependent. In particular, when the two assets are currencies, the best policy is neither fixed nor floating exchange rates; when the two assets are currency and bonds, the best policy is a state-dependent

[1]Aliprantis et al. [1] show that money is not essential in the Lagos-Wright [5] model. However, Zhu [16] shows that their argument is not robust to a commodity-money refinement—attaching an arbitrarily small utility payoff to money, one that in the limit plays no role in the monetary equilibrium.

discount (or premium) on the bonds.

It turns out that we are able to support the outcome that is best if there were perfect monitoring—memory. Notably, even with memory, the presence of the linear stage-1 good is crucial for attaining the best allocation. We achieve that optimum when there is no memory by having a particular active policy at stage-1 and a particular stage-2 trading protocol.

The active stage-1 policy must achieve the correct real value of assets in the centralized meeting at each state—the value that equals the amount of utility that memory can transfer at that state. Given no asset-substitution, it must also achieve the correct split of that real value between the two assets and the correct (expected) rate of return of each asset. The split is about risk sharing. Each asset's rate of return and its real value determine the asset's (expected) future value, a value that is used to compensate the current production cost in the sector's stage-2 meetings. The (across-state) splits and the planner's asset-supply policy together determine the rates of return. Our main contribution is showing that there exist optimal splits and asset-supply policy that support the full-memory optimum and that the asset-supply policy has the features mentioned above.

The stage-2 trading protocol is a generalization of that in Hu et al. [2]. In a one-asset and one-sector version of the model without shocks, they show that there is a protocol which, by rewarding buyers who hold a sufficient quantity of money, achieves the outcome that would be best in a version of the model with memory. Our protocol is similar except that it also uses the idea in Zhu and Wallace [17] that endogenizes a favored asset for each sector.[2]

Our results depend on the quasi-linear preference feature of the Lagos-Wright [5] model, the feature that implies that the distribution of wealth entering a date is not a state variable of the model. In the conclusion, we say why we expect similar results for policy to hold in less extreme versions of the model, versions in which the linear good appears only periodically or not at all.

[2] Rocheteau and Nosal [9] use the protocol of Zhu and Wallace [17] in a two-country non-stochastic version of the model to get a determinate exchange rate. Hu and Rocheteau [3] use versions of the protocols in Zhu and Wallace [17] and Hu et al. [2] to avoid capital over-accumulation in a version of the Lagos-Wright [5] model. The same class of protocols (from Zhu and Wallace [17] and Hu et al. [2]) is used in Zhu [15], which applies the present model to compare fixed and flexible exchange rates.

2 The environment

Time is discrete, dated as $t \geq 0$. There is a [0,2] continuum of infinitely-lived agents. The economy has two sectors, 1 and 2. At the start of period-0, agents are randomly assigned to a sector and keep the sector affiliation forever. Each period has two stages. Everyone is together at stage-1, while stage-2 has two-person meetings in each sector. Stage-2 has one produced and perishable good in each sector. At stage-1, each agent can produce and consume a good, whose payoff in terms of utility is equal to the amount consumed or produced (positive for consumption, negative for production). This is the critical feature of the Lagos-Wright [5] model that makes the model tractable. (Notice that there are no aggregate gains from trade at stage-1.)

At the beginning of stage-1 of each period, two shocks are realized. Each agent gets an idiosyncratic shock which makes them either a stage-2 producer or a consumer for one date with equal probability.[3] And there is a publicly observed aggregate shock which determines stage-2 preferences and technologies. There are I aggregate states. The transition of states follows a Markov chain with a transition matrix $\pi = (\pi_{ij})$, which has a unique invariant distribution $\alpha = (\alpha_1, ..., \alpha_I)$. Let $i \in \{1, ..., I\}$ be the current (aggregate) state. Then in sector-h, $h \in \{1, 2\}$, the utility of a consumer who consumes $q \geq 0$ is $U_h(q, \theta_{ih})$ and the disutility of a producer who produces q is $C_h(q, \rho_{ih})$, where $\theta_{ih}, \rho_{ih} > 0$, $U_h(0, \theta_{ih}) = C_h(0, \rho_{ih}) = 0$, $U_{hq} \equiv \partial U_h / \partial q > 0$, $\partial U_{hq} / \partial q < 0$, $C_{hq} \equiv \partial C_h / \partial q > 0$, $\partial C_{hq} / \partial q \geq 0$, $\beta U_{hq}(0, \theta_{ih}) > C_{hq}(0, \rho_{ih})$, and $\beta \in (0, 1)$ is the discount factor. Each agent maximizes expected discounted utility with period utility given by the sum of stage-1 utility and stage-2 utility.

There are two outside nominal assets. We consider two settings: (a) the two assets are two currencies; and (b) asset-1 is currency and asset-2 is a nominal one-period discount government bond. Most of the analysis is common to both settings.

3 Mechanisms and equilibrium

There is a planner whose objective is to maximize *ex ante* welfare, each agent's expected discounted utility prior to the realization of any shock at period-0. Because of linearity of stage-1 goods, *ex ante* welfare only depends on stage-2 output. In

[3] In earlier versions, there was a second idiosyncratic shock which determined whether a consumer visited the non-home sector at stage-2 for one period. It turned out that such a shock plays no role.

particular, $W(y) = \sum_{(i,h)} \alpha_i[U_h(y_{ih}, \theta_{ih}) - C_h(y_{ih}, \rho_{ih})]$ is the welfare value of an *allocation* $y = \{(y_{i1}, y_{i2}) : 1 \leq i \leq I\}$, where y_{ih} is output in a stage-2 meeting in sector-h at state-i. The planner is subject to the constraint that all trades be in the meeting-specific core.

Because all agents meet together at stage-1, the relevant core is degenerate and the planner's choice effectively comes down to asset-supply policy. Such policy could either be described in terms of quantities or in terms of prices. We choose prices so that the planner stands ready to exchange one asset for the other at a (state-dependent) price. This price is the nominal exchange rate in setting (a) and the nominal interest rate in setting (b). Agents trade the linear good for each asset in a respective Shapley-Shubik [10] trading post; they can trade one asset for the other with the planner before and after trading at the two posts. The realized price in each post is common knowledge. The individual bids at each post and the individual asset trades are private information. There is no explicit taxation.[4]

Because agents meet in pairs at stage-2, the core for each pairwise meeting is non-degenerate (provided that the consumer in a meeting holds valued assets) and the planner's choice is a trading protocol that assigns a trading outcome from the pairwise core for each meeting. The game form for a pairwise meeting has two rounds of moves. In the first round, the two agents say *yes* or *no* simultaneously. If both say *yes*, then the meeting moves to the second round; otherwise, the meeting ends up with autarky. In the second round, the consumer proposes a trading outcome and then the producer says *yes* or *no*. The game ends up with the proposed outcome if the producer says *yes* and with the outcome assigned by the trading protocol otherwise.[5] Each agent can observe his meeting partner's portfolio, and the actions taken in the meeting are only observed by the two agents in the meeting.

A *mechanism* is therefore represented by a stage-1 asset-supply policy and a stage-2 trading protocol. A mechanism is a *Markov mechanism* if dependence of the stage-1 policy and the stage-2 trading protocol on history is restricted to the current state. To be precise, let i be the current state. The planner stands ready to exchange ξ_i units of asset-1 for one unit of asset-2 so that $\xi = (\xi_1, ..., \xi_I)$ represents a Markov asset-supply policy. For a stage-2 trading protocol, consider a meeting in sector-h at state-i. En-

[4]As established in Hu et al. [2], lump-sum taxation, of the sort that would support the Friedman rule, is not feasible in the model.

[5]This game form is used in Hu et al. [2] and Zhu [14].

tering the meeting, the consumer carries a *real portfolio* $x = (x_1, x_2)$ of assets and the producer carries $x' = (x'_1, x'_2)$, where x_k and x'_k are values of asset-k, $k \in \{1, 2\}$, in terms of the stage-1 good in the current stage-1 market. Then the trading outcome assigned by the trading protocol is $\mu_{ih}(x, x') = (y_{ih}(x, x'), p_{ih1}(x, x'), p_{ih2}(x, x'))$, where $y_{ih}(x, x')$ is the producer's output (and the consumer's consumption) and $p_{ihk}(x, x')$ is the consumer's payment in asset-k. Let μ_{ih} denote the mapping $(x, x') \mapsto \mu_{ih}(x, x')$ so that $\mu = \{(\mu_{i1}, \mu_{i2}) : 1 \leq i \leq I\}$ represents a Markov trading protocol, and (ξ, μ) represents a Markov mechanism. We focus on Markov mechanisms.

Given a mechanism (ξ, μ), an agent's strategy specifies the stage-1 actions and the stage-2 actions for each history. An agent's strategy is a *Markov strategy* if when the realizations of the aggregate shock and the agent's idiosyncratic shock are the same in any two periods t and t', it results in (i) the same end-of-stage-1 real portfolio in t and t' provided that the realized prices at trading posts imply the same relative prices of two assets in t and t', and (ii) the same stage-2 actions in t and t' provided that the agent and his meeting partner carry the same real portfolios and his partner take the same actions in t and t'. We focus on Markov strategies.

Definition 1 *Given a mechanism (ξ, μ), a profile of strategies is an* equilibrium *if each strategy in the profile is a Markov strategy and the profile evaluated at any history is a Nash equilibrium.*

Definition 2 *An allocation y is supported by a mechanism (ξ, μ) if, given (ξ, μ), there exists an equilibrium whose outcome is y; it is an equilibrium allocation if it is supported by some mechanism; and it is optimal if its welfare value $W(y)$ is highest among all equilibrium allocations. A mechanism is optimal if it supports an optimal equilibrium allocation; an asset-supply policy is optimal if it is part of an optimal mechanism.*

4 A candidate optimal equilibrium allocation

We start by describing a set of allocations, denoted Y. The unique allocation that maximizes $W(y)$ on Y is the candidate optimal equilibrium allocation. The construction of the set Y uses a minimal condition for an allocation to be an equilibrium allocation: the economy-wide future net utility gains must be sufficient to cover the current stage-2 utility cost of production.

Proposition 1 *For an allocation y, let $u_{ih}(y) = U_h(y_{ih}, \theta_{ih})$, $c_{ih}(y) = C_h(y_{ih}, \rho_{ih})$, $d_{ih}(y) = u_{ih}(y) - c_{ih}(y)$, $c_i(y) = 0.5[c_{i1}(y) + c_{i2}(y)]$, and $d_i(y) = 0.5[d_{i1}(y) + d_{i2}(y)]$. Let $\kappa_i(y) = [\sum_{t\geq 1}\sum_j \beta^t \pi_{ij}(t) d_j(y)] - c_i(y)$, where $\pi_{ij}(t)$ is the t-step transition probability from state i to j. Let $Y = \{y : \kappa_i(y) \geq 0 \text{ all } i\}$.*

(i) If y is an equilibrium allocation, then $y \in Y$.

(ii) There exists a unique solution, denoted y°, to the problem $\max_{y\in Y} W(y)$.

(iii) Let y^* be the efficient allocation, i.e., $U_{hq}(y^*_{ih}, \theta_{ih}) = C_{hq}(y^*_{ih}, \rho_{ih})$ all (i, h). Then $y^\circ \leq y^*$.

(iv) Let $y_i = (y_{i1}, y_{i2})$. For each i, $\kappa_i(y^\circ) > 0$ only if $y^\circ_i = y^*_i$.

(v) For each i, $U_{2q}(y^\circ_{i2}, \theta_{i2})/C_{2q}(y^\circ_{i2}, \rho_{i2}) = U_{1q}(y^\circ_{i1}, \theta_{i1})/C_{1q}(y^\circ_{i1}, \rho_{i1})$.

Proof. For part (i), suppose y is an equilibrium allocation. When the current period is t and the current state is i, let $N_{iht}(0)$ and $N_{iht}(1)$ denote the sets of agents who consume and produce y_{ih} at stage-2 in sector-h, respectively. Let $v_{iht}(n, s)$ denote the continuation value of agent n at the end of period t if $n \in N_{iht}(s)$ for $s \in \{0, 1\}$. Let $V_{it}(s) = \int_{n\in N_{iht}(s)} [v_{i1t}(n, s) + v_{i2t}(n, s)]dn$. Then the economy-wide continuation value at the end of period t is

$$V_{it} = V_{it}(0) + V_{it}(1). \tag{1}$$

Because the economy-wide stage-1 net utility gain in period $t + 1$ is zero, we have $V_{it} = \beta E_i[d_i(y) + V_{jt+1}]$. By iteration, this gives rise to

$$V_{it} = \sum_{\tau\geq 1}\sum_j \beta^\tau \pi_{ij}(\tau) d_j(y). \tag{2}$$

Because each agent can always choose to stay in permanent autarky, his continuation value at the end of t is bounded below by zero for any portfolio held. So $V_{it}(0) \geq 0$. This and (1) imply $V_{it}(1) \leq V_{it}$. Also, in order to ensure that producer $n \in N_{iht}(1)$ in sector-h produces y_{ih} (instead of choosing autarky in the current stage-2), $v_{hit}(n, 1)$ must be bounded below by $c_{ih}(y)$, implying $c_i(y) \leq V_{it}(1)$. Using this, $V_{it}(1) \leq V_{it}$, and (2), we conclude that $y \in Y$. Now consider part (ii). Because $\kappa_i(y)$ is strictly concave in y, Y is convex. Because $W(y)$ is strictly concave in y, $W(y)$ has a unique maximum on Y. Parts (iii) and (iv) are obvious. For part (v), let $\mathcal{I} = \{i : \kappa_i(y^\circ) = 0\}$ and $Y' = \{y \in \mathbb{R}^{2I}_+ : y_i = y^*_i \text{ if } i \notin \mathcal{I}\}$. By part (iv), y° maximizes $W(y) + \sum_{i\in\mathcal{I}} \iota_i \kappa_i(y)$ for $y \in Y'$ (ι_is are Lagrange multipliers), implying the desired property of y°. ■

Using the argument in the proof of Proposition 1 (i), we see that $W(y^\circ) > W(y)$

for any equilibrium allocation $y \neq y^{\circ}$ even when there is *memory*—everyone's identity, type realization in each period, and actions taken at any time are public information.

5 A symmetric two-state example

Here we use a simple example to illustrate how memory supports y° as an equilibrium allocation, and then draw from this a few general lessons regarding how assets may support y°.

Let $I = 2$, $(\theta_{11}, \rho_{11}, \theta_{12}, \rho_{12}) = (\theta_{22}, \rho_{22}, \theta_{21}, \rho_{21})$, $U_1(q, \theta) = U_2(q, \theta)$, $C_1(q, \rho) = C_2(q, \rho)$, and $\pi_{ij} = 0.5$ all (i, j). That is, sector-1 in state-1 is like sector-2 in state-2, and vice versa; and each state occurs with probability 0.5 in each period. In particular, there are sector-specific risks but there is no aggregate risk. To further simplify matters, let $y^{\circ} = y^*$. Denote by d^* the common value of $d_1(y^*)$ and $d_2(y^*)$, and by c^* the common value of $c_1(y^*)$ and $c_2(y^*)$. Then $\sum_{t \geq 1} \sum_j \beta^t \pi_{ij}(t) d_j(y^*) = \beta(1-\beta)^{-1} d^*$ so that $y^{\circ} = y^*$ implies $\beta(1-\beta)^{-1} d^* \geq c^*$. Let $u^*_{ih} = u_{ih}(y^*)$, $c^*_{ih} = c_{ih}(y^*)$, and $d^*_{ih} = d_{ih}(y^*)$. Suppose $c^*_{11} > c^*_{12}$ and that β is such that

$$\beta(1-\beta)^{-1} d^* = c^*. \tag{3}$$

Suppose there is memory (and no nominal assets). It is useful to start with the risk-sharing role of stage-1. If there were no stage-1, then y^* could not be an equilibrium allocation. For, the planner could only rely on sector-1 future net utility gains $0.5\beta(1-\beta)^{-1}(d^*_{11} + d^*_{21}) = \beta(1-\beta)^{-1} d^*$ (note $d^*_{21} = d^*_{12}$) to compensate the stage-2 production cost c^*_{i1} in sector-1. This requires

$$\beta(1-\beta)^{-1} d^* \geq \max\{c^*_{11}, c^*_{21}\} = \max\{c^*_{11}, c^*_{12}\} = c^*_{11}.$$

But by (3), $c^*_{11} > c^*_{12}$ and $c^* = 0.5(c^*_{11} + c^*_{12})$ imply $\beta(1-\beta)^{-1} d^* < c^*_{11}$.

Stage-1 provides the simplest way—transferable utility—to achieve risk sharing. When the current period is not the first period, the current state is i, and the last state is l, the planner can ask each sector-h consumer to surrender

$$d^*_{ih} + (c^*_{ih} - c^*) + \beta(1-\beta)^{-1} d^*$$

units of goods at stage-1, and transfer $\beta^{-1} c^*_{lh}$ to each sector-h agent who produced y^*_{lh} in the last period. Consequently, the planner receives $(1-\beta)^{-1} d^*$ and gives out $\beta^{-1} c^*$. By (3), these two amounts are equal. Any consumer who fails to surrender the suitable amount at stage-1 and any producer who fails to produce the suitable amount at stage-2 are punished by permanent autarky. With this arrangement, a sector-h producer's continuation value right after producing y_{ih}^* is $\beta[\beta^{-1}c_{ih}^* - 0.5(1-\beta)^{-1}d^* + (1-\beta)^{-1}d^*] = c_{ih}^* + 0.5c^*$. Thus, permanent autarky is a sufficient threat to ensure that y_{ih}^* is produced at stage-2. A sector-h consumer's continuation value right after receiving y_{ih}^* at stage-2 is $\beta[-0.5(1-\beta)^{-1}d^* + (1-\beta)^{-1}d^*] = 0.5c^*$, implying that the continuation value when exiting stage-1 is $u_{ih}^* + 0.5c^*$. Because

$$d_{ih}^* + (c_{ih}^* - c^*) + \beta(1-\beta)^{-1}d^* = u_{ih}^* - c^* + \beta(1-\beta)^{-1}d^* = u_{ih}^*,$$

permanent autarky is a sufficient threat to ensure that u_{ih}^* is surrendered at stage-1.

This simple risk-sharing arrangement reveals two general lessons. First, in order to cover the production cost of the allocation at each state, the planner must extract at each state at stage-1 the aggregate amount that matches the current and future net utility gains of the allocation. Second, the planner needs to split the aggregate amount between two sectors in a way that satisfies the individual participation constraint at stage-1 in each sector. In the above example, at state-i, the aggregate extracted amount is $(1-\beta)^{-1}d^*$, which is split into $0.5u_{i1}^*$ and $0.5u_{i2}^*$. Needless to say, these two conditions also apply when allocations are achieved using assets instead of memory.

With nominal assets, however, there is an additional constraint. Suppose, as will be the case, that asset-h is only held by sector-h consumers and that each sector-h consumer spends u_{ih}^* to obtain asset-h at stage-1. The additional constraint is that the payment of asset-h made by a consumer at stage-2 must be sufficient to cover the production cost c_{ih}^*. But that depends on the expected rate of return of asset-h. Those rates turn out to depend on how the aggregate extracted amounts are split and on the asset-supply policy.

Now let us use the above discussion to see why some extreme policies may not be optimal. With two currencies, the extreme policies are the policy that results in floating exchange rates and the policy with fixed exchange rates. As shown in Zhu [15], flexible exchange rates eliminate any across-sector risk sharing. Specifically, as with memory but without stage-1, the planner can only rely on each sector's own future net utility gains to compensate its current production cost. Fixed exchange rates, while allowing for risk sharing, introduce a different problem. For general preferences, the fixed rate prevents currencies from being priced according to state-specific fundamentals in the stage-1 market. Consequently, the planner cannot fully utilize the economy-wide current and future net utility gains. In other words, with no room for different assets to have different state-contingent rates of return, a fixed exchange-rate regime cannot generate the pattern of returns necessary to implement the optimal equilibrium allocation. This logic extends to the setting with currency and bonds. With currency and bonds, the extreme policies are the policy that results in a fixed currency-to-bonds ratio (across all states) and the policy that fixes the price of bonds at unity (a zero net interest rate). The latter policy resembles fixed exchange rates. The former resembles flexible exchange rates.

But what is the optimal policy among a continuum of policies between the extremes? Our main contribution is finding a way that jointly determines the optimal (across-state) splits and policy, the splits and policy that support the allocation y°.

6 Rates of return of assets

Given the significance of rates of return of assets discussed above, our task here is to express those rates in an equilibrium (for a given mechanism) by a pair of state-dependent vectors.

Lemma 1 *Given a mechanism (ξ, μ), suppose there exists an equilibrium.*

(i) The ratio of the nominal quantities of the two assets depends on the current state, but not on the date. Moreover, letting λ_i denote the state-i ratio of the nominal quantity of asset-2 to the nominal quantity of asset-1, and z_i denote the state-i amount of stage-1 goods used to acquire all the assets held at the end of stage-1, we have

$$z_{i1} = \frac{z_i}{1+\lambda_i\xi_i} \text{ and } z_{i2} = \frac{\lambda_i\xi_i z_i}{1+\lambda_i\xi_i}, \tag{4}$$

where z_{ik} is the state-i amount of stage-1 goods used to acquire asset-k.

(ii) The expected gross rate of return of each asset depends on the current state but not on the date. Moreover, letting γ_{ik} denote the state-i expected gross rate of return of asset-k, we have

$$\gamma_{i1} = E_i \frac{z_j}{z_i}\frac{1+\lambda_i\xi_i}{1+\lambda_i\sigma_j} \text{ and } \gamma_{i2} = E_i \frac{z_j}{z_i}\frac{1+\lambda_i\xi_i}{1+\lambda_i\sigma_j}\frac{\sigma_j}{\xi_i}, \tag{5}$$

where $\sigma_j = 1$ in setting (a) (assets are two currencies) and $\sigma_j = \xi_j$ in setting (b) (assets are currency and bonds).

(iii) When i is the current state, the continuation value $\nu_{ih}(x)$ for a sector-h agent holding the real portfolio x at the end of the period is $\nu_{ih}(x) = \beta \sum_k x_k \gamma_{ik} + \beta E_i A_{jh}$, where A_{jh} is a constant that does not depend on x.

Proof. When t is the current period and i is the current state, let m^t_{ik} be the nominal quantity of asset-k at the end of stage-1, and let ϕ^t_{ik} be the per unit price of asset-k in units of the stage-1 good at the trading post that trades asset-k and the stage-1 good. Thus, $\phi^t_{ik} m^t_{ik}$ is the total amount of stage-1 goods used to acquire asset-k at stage-1. Recall that the planner stands to exchange one unit of asset-2 for ξ_i units of asset-1. By no-arbitrage at stage-1, $\xi_i = \phi^t_{i2}/\phi^t_{i1}$ so

$$\phi^t_{i1} m^t_{i1} = \frac{\phi^t_{i1} m^t_{i1} + \phi^t_{i2} m^t_{i2}}{1 + (m^t_{i2}/m^t_{i1})\xi_i}. \tag{6}$$

By Definition 1, $\phi^t_{ik} m^t_{ik}$ does not depend on t (strategies are Markov strategies). So by (6), the ratio m^t_{i2}/m^t_{i1} of the nominal quantities of two assets does not depend on t. By the definition of z_i, $z_i = \phi^t_{i1} m^t_{i1} + \phi^t_{i2} m^t_{i2}$. Thus by (6) and the definition of λ_i,

$$m^t_{i1}\phi^t_{i1} = \frac{z_i}{1 + \lambda_i \xi_i} \text{ and } m^t_{i2}\phi^t_{i2} = \frac{\lambda_i \xi_i z_i}{1 + \lambda_i \xi_i}. \tag{7}$$

This proves part (i). When the next state is j, the nominal value of all assets in units of asset-1 in the coming stage-1 market is $m^t_{i1} + m^t_{i2}\sigma_j = m^t_{i1}(1 + \lambda_i \sigma_j)$ and the per unit price of asset-1 is

$$\phi^{t+1}_{j1} = \frac{z_j}{m^t_{i1} + m^t_{i2}\sigma_j} = \frac{z_j}{m^t_{i1}(1 + \lambda_i \sigma_j)}. \tag{8}$$

By (7) and (8),

$$E_i \frac{\phi^{t+1}_{j1}}{\phi^t_{i1}} = E_i\left(\frac{z_j}{z_i} \frac{1 + \lambda_i \xi_i}{1 + \lambda_i \sigma_j}\right) \text{ and } E_i \frac{\phi^{t+1}_{j2}}{\phi^t_{i2}} = E_i\left(\frac{z_j}{z_i} \frac{1 + \lambda_i \xi_i}{1 + \lambda_i \sigma_j} \frac{\sigma_j}{\xi_i}\right).$$

Thus, the expected gross rate of return of asset-k at state-i does not depend on the date. In particular, we have (5). This proves part (ii). Part (iii) follows from the linear preference of stage-1 goods. ■

In what follows, we refer to the vectors $\lambda = (\lambda_1, ..., \lambda_I)$ and $z = (z_1, ..., z_I)$ as the *characterizing vectors* of an equilibrium. Relating to the discussion in section 5, z_i represents the amount of goods received by the planner from agents at stage-1. Suppose only consumers hold assets and sector-h consumers only hold asset-h. Then z_{i1} and z_{i2} represent the split of z_i between consumers of the two sectors.

7 Showing that y° is the optimal equilibrium allocation

By Proposition 1, y° is the unique optimal equilibrium allocation if it is an equilibrium allocation. To show that y° is an equilibrium allocation, we proceed in three steps. First, we construct the candidate characterizing vectors, λ and z, and the candidate asset-supply policy ξ, and show that the rates of return and the split of z (into z_{i1} and z_{i2}) implied by those candidates have the desired properties. Second, we propose the candidate trading protocol μ that induces each sector-h consumer to spend $2z_{ih}$ at state-i.[6] Lastly, we use the above candidates to construct a strategy profile and show that given the candidate mechanism, this profile is an equilibrium and that y° is the outcome of this equilibrium. Throughout this section, we let $c_{ih} = c_{ih}(y^\circ)$, $d_{ih} = d_{ih}(y^\circ)$, $c_i = 0.5(c_{i1} + c_{i2})$, and $d_i = 0.5(d_{i1} + d_{i2})$.

7.1 Candidate z, λ, and ξ

The candidate z is determined by

$$z_i = d_i + \sum_{t\geq 1}\sum_j \beta^t \pi_{ij}(t) d_j. \tag{9}$$

This is the natural candidate because $\beta E_i z_j = \beta E_i[d_j + \sum_{t\geq 1}\sum_l \beta^t \pi_{jl}(t) d_l]$ or

$$\beta E_i z_j = \sum_{t\geq 1}\sum_j \beta^t \pi_{ij}(t) d_j. \tag{10}$$

That is, z fully utilizes the current and future net utility gains of the allocation y°.

The candidate λ and ξ are determined by

$$\frac{c_{i2}}{c_{i1}} = (E_i \frac{z_j}{1+\lambda_i\sigma_j})^{-1} E_i \frac{\lambda_i \sigma_j z_j}{1+\lambda_i\sigma_j}, \tag{11}$$

and

$$\lambda_i \xi_i = (0.5 d_{i1} + \beta E_i \frac{z_j}{1+\lambda_i\sigma_j})^{-1}(0.5 d_{i2} + \beta E_i \frac{\lambda_i\sigma_j z_j}{1+\lambda_i\sigma_j}). \tag{12}$$

Conditions (11) and (12) represent two optimality principles. The first is that the ratio of the (expected) future real values of the two assets is proportional to the ratio

[6] Each individual consumer takes aggregates such as z_i and its split as given when choosing his real portfolio x at stage-1. The optimal x must agree with the aggregates in equilibrium. In particular, if each sector-h consumer chooses $x = x_{ih}$ at state-i, then $x_{ihh} = 2z_{ih}$.

of the current production costs. The second is that the ratio of the current real value of the two assets is proportional to the ratio of the continuation payoffs of agents who hold the two assets. Of course, we need to show that there exists such (λ, ξ).

Lemma 2 *There exists (λ, ξ) so that (11) and (12) hold for all i.*

Proof. For setting (b) (currency and bonds), existence is obvious. Indeed, with $\sigma_j = 1$, (11) and (12) become

$$\lambda_i = \frac{c_{i2}}{c_{i1}} \tag{13}$$

and

$$\lambda_i \xi_i = (0.5 d_{i1} + \frac{\beta E_i z_j}{1+\lambda_i})^{-1}(0.5 d_{i2} + \frac{\lambda_i \beta E_i z_j}{1+\lambda_i}). \tag{14}$$

For setting (a) (two currencies), $\sigma_j = \xi_j$ so (11) and (12) become

$$c_{i2} E_i \frac{z_j}{1+\lambda_i \xi_j} = c_{i1} E_i \frac{\lambda_i \xi_j z_j}{1+\lambda_i \xi_j} \tag{15}$$

and

$$\lambda_i \xi_i = (0.5 d_{i1} + \beta E_i \frac{z_j}{1+\lambda_i \xi_j})^{-1}(0.5 d_{i2} + \beta E_i \frac{\lambda_i \xi_j z_j}{1+\lambda_i \xi_j}). \tag{16}$$

Let $E_i z_j (1+\lambda_i \xi_j)^{-1} = a E_i z_j$ and $E_i \lambda_i \xi_j z_j (1+\lambda_i \xi_j)^{-1} = (1-a) E_i z_j$. By (15), $a c_{i2} = (1-a) c_{i1}$. So $a = c_{i1}(2c_i)^{-1}$ and we have

$$\beta E_i \frac{z_j}{1+\lambda_i \xi_j} = \frac{c_{i1} \beta E_i z_j}{2 c_i} \tag{17}$$

and

$$\beta E_i \frac{\lambda_i \xi_j z_j}{1+\lambda_i \xi_j} = \frac{c_{i2} \beta E_i z_j}{2 c_i}. \tag{18}$$

By (17) and (18), (16) can be written as $\lambda_i = g_i/\xi_i$, where

$$g_i = \frac{d_{i2} + (c_{i2}/c_i)\beta E_i z_j}{d_{i1} + (c_{i1}/c_i)\beta E_i z_j}.$$

Substituting $\lambda_i = g_i/\xi_i$ into (15) yields

$$E_i \frac{c_{i1}\xi_j g_i - c_{i2}\xi_i}{\xi_j g_i + \xi_i} z_j = 0. \tag{19}$$

To show that there exists ξ satisfying (19), let $\Delta = \{\delta = (\delta_1, ..., \delta_I) \in \mathbb{R}^I_+ : \sum_{i=1}^I \delta_i = 1\}$ and for $\varphi \in [0,1]$, define a mapping $L^\varphi = (L_i^\varphi, ..., L_i^\varphi)$ on Δ by

$$L_i^\varphi(\delta) = \varphi[\delta_i + K_i(\delta)] + (1-\varphi)\bar{\delta}_i,$$

where $\bar{\delta}$ is an interior point of Δ (i.e., $\bar{\delta}_i > 0$ all i) and

$$K_i(\delta) = E_i \frac{c_{i1}\delta_j g_i - c_{i2}\delta_i}{\delta_j g_i + \delta_i} z_j.$$

It suffices to show that L^1 has a fixed point. We claim that for any φ, the mapping L^φ does not have a fixed point on the boundary of Δ, i.e., if $\delta \in \Delta$ has $\delta_i = 0$ for some i, then $\delta \neq L^\varphi(\delta)$. To see this, note that if $\delta_i = 0$ then $K_i(\delta) > 0$, which implies $L_i^\varphi(\delta) > 0$. It follows from the claim that L^1 and L^0 have the same fixed-point index (see Zeidler [13, Theorem 12.A, p. 535]). Because that index is 1 for L^0 (see Zeidler [13, Definition 12.3.(A1), p. 529]), L^1 has a fixed point (see Zeidler [13, Proposition 12.4(2), p. 530]). This completes the proof. ■

Lemma 3 *Let (z, λ, ξ) be as specified in (9), (11), and (12). Let z_{ih} and γ_{ih} be as specified in (4) and (5).*

(i) For each (i, h), $\beta z_{ih}\gamma_{ih} \geq 0.5c_{ih}$ and strictly only if $y_i^\circ = y_i^*$.

(ii) For each (i, h), $z_{ih} = 0.5d_{ih} + \beta z_{ih}\gamma_{ih}$.

Proof. Fix i. As shown in the proof of Lemma 2, (11) implies

$$\beta E_i \frac{z_j}{1 + \lambda_i\sigma_j} = \frac{c_{i1}\beta E_i z_j}{2c_i} \text{ and } \beta E_i \frac{\lambda_i\sigma_j z_j}{1 + \lambda_i\sigma_j} = \frac{c_{i2}\beta E_i z_j}{2c_i}. \tag{20}$$

Using $y^\circ \in Y$ and (10), we have $\beta E_i z_j \geq c_i$. So (20) implies

$$\beta E_i \frac{z_j}{1 + \lambda_i\sigma_j} \geq 0.5c_{i1} \text{ and } \beta E_i \frac{\lambda_i\sigma_j z_j}{1 + \lambda_i\sigma_j} \geq 0.5c_{i2}, \tag{21}$$

which by (4) and (5) imply $\beta z_{ih}\gamma_{ih} \geq 0.5c_{ih}$ all h. Now suppose $\beta z_{ih}\gamma_{ih} > 0.5c_{ih}$ for some h. Then $\beta E_i z_j > c_i$, which by Proposition 1 (iv) and (10), implies $y_i^\circ = y_i^*$. To continue, using (12), we have

$$(1 + \lambda_i\xi_i)(0.5d_{i1} + \beta E_i \frac{z_j}{1 + \lambda_i\sigma_j}) = 0.5d_{i1} + \beta E_i \frac{z_j}{1 + \lambda_i\sigma_j} + 0.5d_{i2} + \beta E_i \frac{\lambda_i\sigma_j z_j}{1 + \lambda_i\sigma_j} \tag{22}$$

and

$$\frac{1 + \lambda_i\xi_i}{\lambda_i\xi_i}(0.5d_{i2} + \beta E_i \frac{\lambda_i\sigma_j z_j}{1 + \lambda_i\sigma_j}) = (1 + \lambda_i\xi_i)(0.5d_{i1} + \beta E_i \frac{z_j}{1 + \lambda_i\sigma_j}). \tag{23}$$

Because the right-hand side of (22) is equal to $d_i + \beta E_i z_j$, it follows from (9), (22), and (23) that

$$\frac{z_i}{1 + \lambda_i\xi_i} = 0.5d_{i1} + \beta E_i \frac{z_j}{1 + \lambda_i\sigma_j} \tag{24}$$

and

$$\frac{\lambda_i \xi_i z_i}{1+\lambda_i \xi_i} = 0.5 d_{i2} + \beta E_i \frac{\lambda_i \sigma_j z_j}{1+\lambda_i \sigma_j}, \tag{25}$$

which by (4) and (5) imply $z_{ih} = 0.5 d_{ih} + \beta z_{ih} \gamma_{ih}$ all h. ■

Lemma 3 (i) and (ii) establish two desired properties of the rates of return and the split of z (into z_{i1} and z_{i2}) implied by (z, λ, ξ) in (9), (11), and (12): the future value of asset-h can cover the current stage-2 production cost at sector-h; and the split can satisfy the individual consumer's participation constraint at stage-1.

7.2 Candidate trading protocol μ

The construction of the candidate trading protocol μ uses the rates of return γ_{i1} and γ_{i2} and the asset's real stage-1 values z_{i1} and z_{i2} implied by (z, λ, ξ) in (9), (11), and (12). As it turns out, when assets have those implied rates of return, μ induces a sector-h consumer to leave stage-1 with only asset-h whose real value is $2z_{ih}$ at state-i. It is convenient to describe μ in terms of the following problem.

Problem 1 The meeting outcome $\mu_{ih}(x, x') = (y_{ih}(x, x'), p_{ih1}(x, x'), p_{ih2}(x, x')))$ assigned by μ for a pairwise meeting in sector-h at state-i between a consumer carrying the real portfolio $x = (x_1, x_2)$ and a producer carrying the real portfolio $x' = (x'_1, x'_2)$ is determined by a two-step optimization problem.

Step 1. Let a meeting outcome $(\tilde{y}_{ih}, \tilde{p}_{ih1}, \tilde{p}_{ih2})$ be determined as follows. Let $\tilde{p}_{ihk} = 0$ for $k \neq h$. If $x_h \geq 2z_{ih}$ then

$$(\tilde{y}_{ih}, \tilde{p}_{ihh}) = \arg\max U_h(q, \theta_{ih}) - \beta p_h \gamma_{ih} \tag{26}$$

subject to $q \geq 0$, $0 \leq p_h \leq x_h$, and $-\rho_{ih} c(q) + \beta p_h \gamma_{ih} \geq 0$; otherwise,

$$(\tilde{y}_{ih}, \tilde{p}_{ihh}) = \arg\max -C_h(q, \rho_{ih}) + \beta p_h \gamma_{ih} \tag{27}$$

subject to $q \geq 0$, $0 \leq p_h \leq x_h$, and $U_h(q, \theta_{ih}) - \beta p_h \gamma_{ih} \geq 0$.

Step 2. Let $\mu_{ih}(x, x') = \arg\max -C_h(q, \rho_{ih}) + \beta(p_1 \gamma_{i1} + p_2 \gamma_{i2})$ subject to $q \geq 0$, $0 \leq p_k \leq x_k$ all k, and $U_h(q, \theta_{ih}) - \beta(p_1 \gamma_{i1} + p_2 \gamma_{i2}) \geq U_h(\tilde{y}_{ih}, \theta_{ih}) - \beta \tilde{p}_{ihh} \gamma_{ih}$.

For Problem 1, suppose that the two agents in the problem are in an equilibrium where the rate of return of asset-h is γ_{ih}. The step-2 optimization ensures that the outcome $\mu_{ih}(x, x')$ is in the pairwise core. This follows because this outcome maximizes the producer's payoff conditional on not making the consumer worse off than the trade $(\tilde{y}_{ih}, \tilde{p}_{ih1}, \tilde{p}_{ih2})$ obtained from the step-1 optimization and because there is no restriction on which asset can be used to make payments. The step-1 optimization serves two purposes. First, by the restriction on the payment, it gives the consumer a reward for offering the *right asset* for the meeting. Indeed, if the consumer does not carry the right asset, then the step-1 outcome is always $(0, 0, 0)$ so that the step-2 optimization gives the producer all the surplus from trade. This aspect of the design of the trading protocol is borrowed from Zhu and Wallace [17] and endogenizes no asset-substitution. Second, the step-1 optimization gives the consumer an incentive to spend $2z_{ih}$ at stage-1 on acquiring nominal asset-h. Indeed, if the consumer carries only the right asset in the proper amount, the step-1 optimization assigns all surplus to the consumer. This design is borrowed from Hu et al. [2] and Hu and Rocheteau [3].[7]

Lemma 4 *Let (z, λ, ξ) be as specified in (9), (11), and (12) and let μ be the trading protocol in Problem 1. Let $x_{i1} = (2z_{i1}, 0)$, $x_{i2} = (0, 2z_{i2})$, and $x'_{i1} = x'_{i2} = (0, 0)$.*

(i) $y_{ih}(x_{ih}, x'_{ih}) = y^{\circ}_{ih}$ and $p_{ihh}(x_{ih}, x'_{ih}) < 2z_{ih}$ only if $y^{\circ}_{ih} = y^{*}_{ih}$.

(ii) For every (i, h),

$$x'_{ih} \in \arg \max_{x'=(x'_1, x'_2)} -\sum_k x'_k - C_h(y_{ih}(x_{ih}, x'), \rho_{ih}) + \beta \sum_k [x_k + p_{ihk}(x_{ih}, x'))]\gamma_{ik}. \quad (28)$$

(iii) For every (i, h),

$$x_{ih} \in \arg \max_{x=(x_1, x_2)} -\sum_k x_k + U_h(y_{ih}(x, x'_{ih}), \theta_{ih}) + \beta \sum_k [x_k - p_{ihk}(x, x'_{ih}))]\gamma_{ik}. \quad (29)$$

Proof. By Lemma 3 (ii), $\beta\gamma_{ih} < 1$. For part (i), let $q = y_{ih}(x_{ih}, x'_{ih})$ and $p_h = p_{ihh}(x_{ih}, x'_{ih})$. By (26), $C_h(q, \rho_{ih}) = \beta p_h \gamma_{ih}$. By $\beta\gamma_{ih} < 1$ and Lemma 3 (i), $U_{hq}(q, \theta_{ih}) \geq C_{hq}(q, \rho_{ih})$ and $p_h < 2z_{ih}$ only if $q = y^{\circ}_{ih} = y^{*}_{ih}$. If $q < y^{*}_{ih}$ then $C_h(q, \rho_{ih}) = 2\beta z_{ih}\gamma_{ih}$, which, by Lemma 3 (i), implies $q = y^{\circ}_{ih}$. By $\beta\gamma_{ih} < 1$, part (ii) is immediate. For part (iii), without loss of generality, let us consider a sector-1 consumer who carries a portfolio $x = (x_1, 0)$ at the end of stage-1, and meets a producer carrying x'_{i1} at stage-2. Let $w(x_1)$ denote the consumer's stage-1 and stage-2 payoff from carrying $x = (x_1, 0)$. By part (i), $y_{i1}(x_{i1}, x'_{i1}) = y^{\circ}_{i1}$ so

$$w(2z_{i1}) = -2z_{i1} + U_1(y^{\circ}_{i1}, \theta_{i1}) + \beta[2z_{i1} - p_{i11}(x_{i1}, x'_{i1})]\gamma_{i1} + A, \quad (30)$$

[7] Problem 1 does not represent an extensive game form with two rounds of alternating offers but it may be understood as a gradual bargaining problem (see O'Neill et al. [8]).

where A is a constant independent of x_1 (see Lemma 1 (iii)) and $2z_{i1} > p_{i11}(x, x'_{i1})$ only if $y^\circ_{i1} = y^*_{i1}$. To proceed, let us consider two cases of $x_1 \neq 2z_{i1}$.

Case 1: $x_1 < 2z_{i1}$. Then $w(x_1) = -x_1 + \beta x_1 \gamma_{i1} + A$. By (30) and $\beta\gamma_{ih} < 1$, $w(x_1) < w(2z_{i1})$.

Case 2: $x_1 > 2z_{i1}$. If $y^\circ_{i1} = y^*_{i1}$ then $w(x_1) = w(2z_{i1}) + (x_1 - 2z_{i1})(-1 + \beta\gamma_{i1})$. By (30) and $\beta\gamma_{ih} < 1$, $w(x_1) < w(2z_{i1})$. Now consider $y^\circ_{i1} \neq y^*_{i1}$. Note that $w(.)$ is concave in $[2z_{i1}, \infty)$ and that there exists some $\bar{x}_1 > 2z_{i1}$ such that

$$w(x_1) = -x_1 + U_1(y_{i1}(x, x'_{i1}), \theta_{i1}) + A$$

with $C_1(y_{i1}(x, x'_{i1}), \rho_{i1}) = \beta x_1 \gamma_{i1}$ for $x_1 \in [2z_{i1}, \bar{x}_1)$. It follows that

$$w'_+(2z_{i1}) = -1 + \beta\gamma_{i1} U_{1q}(y^\circ_{i1}, \theta_{i1}) / C_{1q}(y^\circ_{i1}, \rho_{i1}),$$

where $w'_+(2z_{i1})$ is the right derivative of $w(.)$ at $2z_{i1}$. By Lemma 3, we have $2\beta z_{i1}\gamma_{i1} = C_1(y^\circ_{i1}, \rho_{i1})$ and $2z_{i1} = U_1(y^\circ_{i1}, \theta_{i1})$ so

$$U_{1q}(y^\circ_{i1}, \theta_{i1}) / C_{1q}(y^\circ_{i1}, \rho_{i1}) < U_1(y^\circ_{i1}, \theta_{i1}) / C_1(y^\circ_{i1}, \rho_{i1}) = 1/(\beta\gamma_{i1}),$$

where the inequality uses $U_1(0, \theta_{i1}) = C_1(0, \rho_{i1}) = 0$, concavity of $U_1(., \theta_{i1})$, and convexity of $C_1(., \rho_{i1})$. Thus $w'_+(2z_{i1}) < 0$, implying $w(x_1) < w(2z_{i1})$

To conclude, the portfolio x_{i1} dominates the portfolio $(x_1, 0)$ with $x_1 \neq 2z_{i1}$. By $\beta\gamma_{ih} < 1$, the portfolio $(x_1, 0)$ dominates (x_1, x_2) with $x_2 > 0$ for any x_1. This proves part (iii). ■

7.3 Equilibrium

Let us first use (ξ, λ, z) in (9), (11), and (12) and μ in Problem 1 to construct the strategy profile F in which all strategies are the same. The common strategy f in F specifies the following actions for each agent when the current state is i.

(P) The agent is a producer at stage-2 in sector-h. At stage-1, first adjust the ratio of the nominal quantity of asset-2 to the nominal quantity of asset-1 to λ_i (by exchanging assets with the planner). Next sell all the assets at the trading posts. At round-1 of the stage-2 game, say *yes*; at round-2, say *yes* to a proposal (q, p_1, p_2) iff

$$\beta \sum_k [p_k - p_{ihk}(x_{ih}, x')]\gamma_{ik} \geq C_h(q, \rho_{ih}) - C_h(y_{ih}(x, x'), \rho_{ih}),$$

where x' is the agent's real portfolio and x is his partner's.

(C) The agent is a consumer at stage-2 in sector-h. At stage-1, first adjust the ratio of the nominal quantity of asset-2 to the nominal quantity of asset-1 to λ_i (by exchanging assets with the planner). Next, provided that m_1 is the aggregate nominal quantity of asset-1 at the start of the period, submit orders in the trading posts that result in the real portfolio x_{ih} as if the realized prices of asset-1 and asset-2 were $z_{i1}m_1^{-1}$ and $\xi_i z_{i1} m_1^{-1}$, respectively. At round-1 of the stage-2 game, say *yes*; at round-2, when the agent holds x and his partner holds x', propose $\mu_{ih}(x, x')$.

Lemma 5 *Let (ξ, λ, z) be as specified in (9), (11), and (12) and let μ be the trading protocol in Problem 1. Given the mechanism (ξ, μ), the strategy profile F is an equilibrium whose outcome is y°.*

Proof. Suppose all agents follow f all the time. Let i be the current state. Then, provided that m_1 is the aggregate nominal quantity of asset-1 at the start of the period, the realized prices of asset-1 and asset-2 at trading posts are $z_{i1}m_1^{-1}$ and $\xi_i z_{i1} m_1^{-1}$, respectively. Moreover, the consumers and producers who consume and produce in sector-h enter stage-2 with the portfolios x_{ih} and x'_{ih} (see Lemma 4). Furthermore, when a consumer carrying x meets a producer carrying x' in sector-h at state-i, the outcome $\mu_{ih}(x, x')$ is in the pairwise core. Therefore, provided that all other agents follow f all the time, no agent wants to deviate from f in any stage-2 pairwise meeting, and, in addition, by Lemma 4 (ii) and (iii), no agent wants to deviate from f in stage-1. We conclude that F is an equilibrium. By Lemma 4 (i), the outcome of F is y°. This completes the proof. ■

Proposition 2 *The allocation y° is the unique optimal equilibrium allocation.*

Proof. By Lemma 5, y° is an equilibrium allocation. By Proposition 1 (i) and (ii), y° is the unique optimal equilibrium allocation. ■

8 Optimal asset-supply policy

Let (ξ, λ, z) in (9), (11), and (12) be denoted $(\xi^\circ, \lambda^\circ, z^\circ)$. For some specifications of the aggregate uncertainty and the discount factor, there are multiple optimal asset-supply policies. Indeed, given the aggregate uncertainty, which determines the efficient allocation y^*, if β is sufficiently close to unity, then not only is $y^\circ = y^*$, but because the future net utility gains are more than sufficient to cover the current stage-2 production cost, there are multiple (ξ, λ, z) that differ from $(\xi^\circ, \lambda^\circ, z^\circ)$ and support y°. Our interest is in the optimal policies when the planner is constrained by the future net utility gains. To be precise, we introduce in the following lemma a reference point of the discount factor for an arbitrary y, denoted $\beta(y)$, such that the planner is constrained when $\beta = \beta(y^\circ)$.

Lemma 6 *Let $\beta_i(y)$ be uniquely determined by $\sum_{t\geq 1}\sum_j [\beta_i(y)]^t \pi_{ij}(t) d_j(y) = c_i(y)$. Let $\beta(y) = \max_i \beta_i(y)$.*

(i) $y \in Y$ if and only if $\beta \geq \beta(y)$.

(ii) $\beta > \beta(y^\circ)$ only if $y^\circ = y^*$.

(iii) Let $\beta = \beta(y^\circ)$. Suppose (ξ, μ) supports y° and let λ and z be the characterizing vectors of the corresponding equilibrium. Then

$$\lambda_i \xi_i = \lambda_i^\circ \xi_i^\circ \tag{31}$$

for all i and

$$\frac{c_{l2}(y^\circ)}{c_{l1}(y^\circ)} = (E_l \frac{z_j}{1+\lambda_l \sigma_j})^{-1} E_l \frac{\lambda_l \sigma_j z_j}{1+\lambda_l \sigma_j} \tag{32}$$

for $l \in \arg\max_i \beta_i(y^\circ)$; moreover, if $y_i^\circ \neq y_i^*$ all i, then $(\lambda, \xi) = (\lambda^\circ, \xi^\circ)$.

Proof. For part (i), for the "if" part, suppose $\beta \geq \beta(y)$. Then $\kappa_i(y) \geq 0$ all i so $y \in Y$. For the "only-if" part, suppose $\beta < \beta(y)$. Then $\kappa_i(y) < 0$ for some i so $y \notin Y$. For part (ii), suppose by contradiction that $\beta > \beta(y^\circ)$ while $y^\circ \neq y^*$. Without loss of generality, suppose $y_{11}^\circ < y_{11}^*$. Let y be such that $y_{ih} = y_{ih}^\circ$ if $(i,h) \neq (1,1)$ and $y_{11} = y_{11}^\circ + \epsilon$, where $\epsilon > 0$. By $\beta > \beta(y^\circ)$, $\kappa_1(y^\circ) > 0$ so $y \in Y$ when ϵ is sufficiently small. Clearly, $W(y) > W(y^\circ)$, a contradiction. For part (iii), first note that when $\beta = \beta(y^\circ)$, z must equal z°. Examining (22)-(25) in the proof of Lemma 3, one sees that (31) is necessary for $z = z^\circ$. Also, by $\beta = \beta(y^\circ)$, $c_l(y^\circ) = \beta E_l z_j^\circ$ so (32) must hold. Further suppose $y_i^\circ \neq y_i^*$ all i. Then λ and ξ must satisfy (11) and (12) so $(\lambda, \xi) = (\lambda^\circ, \xi^\circ)$. ■

We say that an asset-supply policy ξ is the simple policy if $\xi_i = 1$ all i. Given an exogenously fixed number $\upsilon > 0$, we say that ξ is a fixed-asset-ratio policy if it happens to result in an equilibrium with $\lambda_i = \upsilon$ all i. For setting (a) (two currencies), the simple policy represents a fixed exchange-rate regime, and a fixed-asset-ratio policy represents a flexible exchange-rate regime.

Proposition 3 *Suppose $\beta = \beta(y^\circ)$. Let $l \in \arg\max_i \beta_i(y^\circ)$.*

(i) Suppose $U_{2q}(y^\circ_{l2}, \theta_{l2})/C_{2q}(y^\circ_{l2}, \rho_{l2}) = U_{1q}(y^\circ_{l1}, \theta_{l1})/C_{1q}(y^\circ_{l1}, \rho_{l1})$ implies

$$U_2(y^\circ_{l2}, \theta_{l2})/C_2(y^\circ_{l2}, \rho_{l2}) \neq U_1(y^\circ_{l1}, \theta_{l1})/C_1(y^\circ_{l1}, \rho_{l1}). \tag{33}$$

Then the simple policy is not optimal.

(ii) Let $C_{h\rho} = \partial C_h/\partial\rho$ and suppose $C_{1\rho}(y^\circ_{l1}, \rho_{l1})C_{2\rho}(y^\circ_{l2}, \rho_{l2}) \neq 0$. Then when the shock vector $\{(\theta_{i1}, \rho_{i1}, \theta_{i2}, \rho_{i2})\}_{i=1}^I$ is outside a measure-zero set in $\mathbb{R}^{4I}$, a fixed-asset-ratio policy is not optimal.

Proof. Suppose (ξ, μ) supports y° and let λ and z be the characterizing vectors of the corresponding equilibrium. For part (i), suppose by contradiction that ξ is the simple policy. Then by (31) and (12),

$$\lambda_l = [0.5d_{l1}(y^\circ) + \beta(y^\circ)E_l\frac{z^\circ_j}{1+\lambda_l}]^{-1}[0.5d_{l2}(y^\circ) + \beta(y^\circ)E_l\frac{\lambda_l z^\circ_j}{1+\lambda_l}]. \tag{34}$$

By (32) and (11),

$$\lambda_l = \frac{c_{l2}(y^\circ)}{c_{l1}(y^\circ)}. \tag{35}$$

By (34), $\lambda_l = d_{l2}(y^\circ)/d_{l1}(y^\circ)$. Then by (35), $d_{l2}(y^\circ)/d_{l1}(y^\circ) = c_{l2}(y^\circ)/c_{l1}(y^\circ)$ or equivalently,

$$U_2(y^\circ_{l2}, \theta_{l2})/C_2(y^\circ_{l2}, \rho_{l2}) = U_1(y^\circ_{l1}, \theta_{l1})/C_1(y^\circ_{l1}, \rho_{l1}). \tag{36}$$

But by Proposition 1 (v), we must have (33), which contradicts (36).

For part (ii), suppose ξ is a fixed-asset-ratio policy. Then (32) becomes

$$c_{l1}(y^\circ)E_l\frac{\upsilon\sigma_j z_j}{1+\upsilon\sigma_j} = c_{l2}(y^\circ)E_l\frac{z_j}{1+\upsilon\sigma_j}$$

for some given $\upsilon > 0$. For setting (b) (currency and bonds), this leads to

$$\upsilon c_{l1}(y^\circ) = c_{l2}(y^\circ), \tag{37}$$

which can only hold for a measure zero set of $\{(\theta_{i1}, \rho_{i1}, \theta_{i2}, \rho_{i2})\}_{i=1}^I$ in $\mathbb{R}^{4I}$. For setting (a), the argument is somewhat indirect. Following the argument in Zhu [15], we can show that with flexible exchange rates, y° is an equilibrium outcome only if

$$c_{lh}(y^\circ) = \beta E_l z_{jh} \tag{38}$$

all h. But again, (38) can only hold for a measure zero set of $\{(\theta_{i1}, \rho_{i1}, \theta_{i2}, \rho_{i2})\}_{i=1}^I$ in $\mathbb{R}^{4I}$. ■

The assumption in Proposition 3 (i) is a specific property of preferences: if the total utility-disutility ratios are equal across sectors in a state, then their marginal ratios must differ. This property ensures that a simple policy cannot be optimal. As shown in the proof of Proposition 3, for the simple policy to implement the optimal allocation, it must align the total ratios. But the optimal allocation itself has aligned marginal rates. Thus, the assumption rules out optimality of the simple policy. It seems that the only specification of U_h and C_h that violates the assumption is multiplicative shocks and power functions; i.e., $U_h(q, \theta) = \theta q^a$ and $C_h(q, \rho) = \rho q^b$.

In the proof of Proposition 3 (ii), (38) is the consequence of flexible exchange rates indicated at the end of section 5. That is, the value of currency h is completely determined by the current and future net utility gains of sector-h and, therefore, sector-h can only rely on its own future net utility gains to compensate its current production cost. With currency and bonds, there is only one nominal asset at the start of each period, and the value of the nominal asset at that time is determined by the economy-wide current and future net utility gains. But because the currency-bond ratio is fixed exogenously, each sector can only use a fixed part of economy-wide future net utility gains, resulting in (37).

9 Concluding remarks

We have found that complicated asset arrangements are helpful for two-sector Lagos-Wright [5] economies subject to sector-specific shocks if the shocks are large enough or if there is sufficient impatience. Suppose we consider a more general model in which the linear good appears only periodically, rather than every period.[8] For a simple policy and given shocks, it is evident that the discount factor consistent with achievement of the efficient outcome increases as the period (between linear good meetings) increases. Thus, it is very likely that the set of parameters for which a simple policy is best shrinks as the period increases. In the limit, when the linear good never appears, it is likely that there is no discount factor consistent with achievement of the efficient outcome (see Wallace [12]). Finally, there is nothing special about two sectors. A plausible surmise is that a version of the model with k sectors is best served if there are k assets with k different state-specific rates of return.

[8] Hu and Rocheteau [3] have shown that the version with period two has a role for private credit and no such role when the linear good appears at every period.

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