Nonneutrality of Exchange-Rate Regimes with Flexible Prices: A Production Channel*

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Abstract

A classic benchmark in international macroeconomics holds that exchange-rate regimes are neutral under flexible prices. This paper overturns the benchmark via a production channel: the current nominal quantity of a currency affects current output through its expected future real value. The channel is illustrated in a two-country Lagos–Wright model with country-specific cash-in-advance constraints. Notably, flexible rates can produce greater real exchange-rate volatility than fixed rates—the Mussa [24] finding. When lump-sum taxes are subject to individual participation constraints, fixed exchange rates dominate if worldwide fundamentals are sufficiently stable across relevant states; in general, neither regime is unambiguously superior.

Keywords: Flexible prices; exchange-rate regimes; unified currency; optimal currency areas; regime neutrality

*An early version of this paper was written jointly under the title "Fixed and Flexible Exchange Rates in Two Matching Models: Non-Equivalence Results." That project studied both a competitive cash-in-advance model and a pairwise-meeting trading environment, with the two formulations delivering the same qualitative message. Differences in views regarding how to proceed with the analysis led to the end of collaboration. The project subsequently developed into a broader working paper under the title "On Nonneutrality of the Exchange-Rate Regime." The analysis related to Mundell's [24] risk-sharing argument for currency unions in that working paper, using the pairwise-meeting trading environment, was later developed separately and appears in Zhu [27]. I am grateful to Neil Wallace for many insightful discussions and suggestions.

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1 Introduction

A classic benchmark in international macroeconomics, originating with Friedman [6, p. 165], holds that exchange-rate regimes are neutral when internal prices are fully flexible, as adjustments can occur equivalently through prices or exchange rates. This regime neutrality—neutrality hereafter—is formalized by Lucas [20] using an endowment economy with country-specific cash-in-advance (CIA) constraints and complete asset markets. Even though the literature has shown that neutrality can fail under incomplete asset markets, it has often served as a reference (Mussa [24]; Obstfeld and Rogoff [25]). This paper shows that, independently of asset-market imperfections and even with fully flexible prices, exchange-rate regimes can differ in their effects on output, real exchange-rate volatility, and welfare once production is elastic. Specifically, elastic production opens a channel, referred to as the production channel, that overturns neutrality.

The channel works as follows. The current nominal quantity of a currency affects current output through its (expected) future real value. Suppose a shock increases demand for dollars. Under the flexible exchange-rate regime—the flexible-rate regime hereafter—the dollar appreciates while quantities remain fixed; under the fixed exchange-rate regime—the fixed-rate regime hereafter—dollar quantity rises and euro quantity falls. For *future neutrality* to hold, the future real value of each currency must be regime-invariant. But then, the higher current dollar quantity under the fixed-rate regime should induce more U.S. production and less EU production, contradicting the presumed *current neutrality*. Apparently, this channel is absent from exchange-rate models with no production (Lucas [20]; the money-in-the-utility-function model of Obstfeld and Rogoff [25, section 8.7]), or with no currencies (Gabaix and Maggiori [7]; Itskhoki and Mukhin [10]).

To formalize the production channel, I employ a two-country Lagos–Wright [18] model similar to the one in Gomis-Porqueras et al. [8] (who focus on quantitative analysis of flexible exchange rates). Each country produces a single good. Prior to trading goods for currencies domestically, people trade currencies and a linear good in a worldwide market to respond to an aggregate shock with country-specific effects; the trade of the linear good plays the role of the asset market in the Lucas [20] model. As in the Lucas [20] model, the trade in each domestic market is subject to the country-specific CIA constraint. Policy takes the simplest form in this pure-currency model:

governments may change quantities of currencies by lump-sum taxes.

I start with a simple example with zero taxes to illustrate the production channel. Remarkably, in the example, the flexible-rate regime can experience greater volatility of the real exchange rate than the fixed-rate regime, suggesting that the renowned finding of Mussa [24] need not validate nominal rigidity but may support nonneutrality with flexible prices. I then establish a general result: nonneutrality holds generically.

Another general result pertains to the optimal regime. When policy operation is costless, the flexible-rate regime can exploit exchange-rate flexibility to mimic any fixed-rate equilibrium. In particular, both regimes can implement the same optimal policy—the Friedman rule, which stabilizes rates of return on currencies to offset discounting and supports the efficient allocation. A novelty of this paper is to introduce an endogenous tax-enforcement constraint. In the model, the most severe punishment for tax rebellion is permanent exclusion from the economy. Therefore, the cost of paying current taxes cannot exceed the continuation payoff for staying. Conditional on universal compliance, the two regimes share the same continuation payoff in every state. When worldwide fundamentals are stable in relevant states, the fixed-rate regime performs better in that it supports the efficient allocation over a wider range of the discount factor. Why? By integrating currency values, the fixed-rate regime avoids unnecessary variation in stabilization costs that arises from independent fluctuations in each currency's value under flexible rates, thereby entailing smaller tax spikes. In general, however, neither regime is unambiguously superior.

In the literature, Helpman and Razin [9] attribute nonneutrality to imperfect capital markets and Lahiri et al. [19] to limited participation in small open economies with flexible prices. Jeanne and Rose [12] highlight noise trading as a source of multiple equilibria under floats. More recently, Itskhoki and Mukhin [11] build regime differences on intermediaries bearing exchange-rate risk. The contribution here is to show that regime-dependent real effects arise directly from monetary responses to aggregate shocks—changes in nominal quantities or exchange rates that are well documented empirically. Exchange-rate arrangements then determine how worldwide fundamentals map into welfare. This link to worldwide fundamentals arguably belongs at the forefront of evaluations of exchange-rate regimes, even when the welfare-driving policy goes beyond a constrained Friedman rule.

In optimal currency area (OCA) theory, Mundell [22] identifies lower transaction costs from a common currency (equivalent to country-specific currencies under fixed rates) as a benefit of monetary integration, and the loss of independent stabilization policy as a cost under sticky prices. Cooper and Kempf [3] revisit this trade-off with flexible prices, modeling multi-currency costs via portfolio-adjustment frictions and policy rigidity through shared inflation taxes. Recent studies explore fiscal coordination in currency unions (Farhi and Werning [5]). My model abstracts from exogenous transaction costs and policy rigidity, and assumes perfect fiscal integration in the sense of Kenen [13]. Yet it generates a regime-dependent endogenous operational cost of monetary policy, stemming from limits to tax enforcement and aggregate shocks.[1] On a general level, such costs add a new factor to the OCA cost–benefit analysis.

2 A two-country variant of the Lagos-Wright model

There are two countries, 1 and 2. Each is populated by a nonatomic unit measure of infinitely-lived people and has its own currency. Each date has two stages; each stage has a produced and perishable good. Actions on a date are depicted as follows:

Individual		Three		**Stage 1**		**Stage 2**
person	$\rightarrow$	shocks	$\rightarrow$	linear good	$\rightarrow$	trade in each
$m = (m_1, m_2)$		realized		and two currencies		country

Specifically, each person enters a date with a portfolio $m = (m_1, m_2) \in \mathbb{R}^2_+$, where m_k is the amount of currency $k \in \{1, 2\}$, i.e., country k's currency. Then, three shocks, two idiosyncratic shocks and one aggregate shock, are realized. One idiosyncratic shock determines a person to be a *producer* or a *consumer* at stage 2 for that date with equal probability. The other determines a constant fraction λ of consumers in each country to be *tourists* at stage 2 for that date—this shock introduces minimal real international trade, though it turns out to have little effect on the main results. The aggregate shock determines the current aggregate state; there are I aggregate states. State transitions follow a Markov chain with a *transition matrix* $\pi = (\pi_{ij})$.

At stage 1, everyone can produce and consume a linear good, i.e., one's utility from consuming q is q and from producing q is $-q$. At stage 2, a producer produces and a nontourist consumer consumes in the home country; a tourist consumes in

[1] A group of papers, including Matsuyama et al [21], Kocherlakota and Krueger [17], Kocherlakota [16], Dong and Jiang [4], Kiyotaki and Moore [14], and Araujo and Ferraris [1], concerns whether one or two currencies better facilitate bilateral trades, but abstracts from aggregate shocks.

the foreign country and returns home at the end of the date. When the current (aggregate) state is $i \in \{1, 2, ..., I\}$, the utility of a consumer who consumes $q \geq 0$ in country k is $u_{ik}(q) \equiv \theta_{ik} u(q)$ and the disutility of a producer who produces q in country k is $c_{ik}(q) \equiv \rho_{ik} c(q)$, where $\theta_{ik}, \rho_{ik} > 0$, $u(0) = c(0) = 0$, $u' > 0$, $u'' < 0$, $c' > 0$, $c'' \geq 0$, $\beta u'(0) > c'(0)$, and β is the discount factor. Each person's period utility is the sum of his stage-1 utility and stage-2 utility; he maximizes expected discounted utility.

I refer to $\alpha \equiv (\theta_{i1}, \rho_{i1}, \theta_{i2}, \rho_{i2})_{i=1}^{I}$ as the *shock vector* and represent the aggregate shock by (π, α). Say a shock (π, α) is *symmetric* if each i has a symmetric state $\sigma(i)$, i.e., there is a mapping $\sigma : \{1, 2, ..., I\} \to \{1, 2, ..., I\}$ such that $\sigma(\sigma(i)) = i$ and $(\alpha_{\sigma(i)1}, \alpha_{\sigma(i)2}) = (\alpha_{i2}, \alpha_{i1})$ all i, $\pi_{ij} = \pi_{\sigma(i)\sigma(j)}$ all (i, j), and $\sigma(i) = i$ at most one i (states come in country-swapped pairs, with at most one self-symmetric state).

At stage 1, everyone can trade the two currencies and the linear good in a worldwide competitive market. The initial stock of each currency is normalized to 0.5. The exchange rate—the relative price of the two currencies—is either flexible or fixed. Under the fixed-rate regime, as is standard, the government of each country is committed to supplying unlimited amounts of its own currency; the fixed value of the exchange rate is normalized to unity.

The governments run state-contingent *policy* $\gamma = (\gamma_{i1}, \gamma_{i2})_{i=1}^{I}$, financed by *country-specific lump-sum* taxes. Let i be the *current* state and s_{ik} be the stock of currency k at the end of the current stage 1. Then the policy withdraws $(1 - \gamma_{ik}) s_{ik}$ units of currency k in the stage-1 market of the *next* period (a negative withdrawal corresponds to a positive injection, and there is no withdrawal in the first period).

At stage 2, each country has a competitive domestic market where the trade of the stage good is subject to the country-specific CIA constraint—buyers must use currency k to purchase goods in country k's market. (The competitive market can be interpreted as the outcome of pairwise trading under a competitive pricing rule, as in Rocheteau and Wright [26].)

3 Equilibrium

To define an equilibrium for a given policy γ, it is convenient to renormalize the stock of each currency at the end of stage 1. Under the flexible-rate regime, the renormalized stock of each currency is set to 0.5; under the fixed-rate regime, the renormalized total stock of the two currencies is set to unity.

I limit consideration to stationary equilibria, in which when i is the current state, the amount of goods ϕ_{ik} paid in the current stage-1 market to obtain one (renormalized) unit of currency k at the end of stage 1 depends only on i.

Let ζ_{ik} denote the (gross expected) rates of return from carrying currency k into the stage-1 market of the next period when the current state is i. Given γ, when j is the next state, under the flexible-rate regime, the per-unit price of currency k in the coming stage-1 market is $(\gamma_{ik})^{-1}\phi_{jk}$ so

$$\zeta_{ik} = (\gamma_{ik})^{-1} E_i(\phi_{jk}/\phi_{ik}); \tag{1}$$

under the fixed-rate regime, the per-unit price of currency k in the coming stage-1 market is $(s_{i1}\gamma_{i1} + s_{i2}\gamma_{i2})^{-1}\phi_{jk}$ (note that $\phi_{j1} = \phi_{j2}$ all j and recall that s_{ik} is the stock of currency k at the end of the current stage 1) so

$$\zeta_{ik} = (s_{i1}\gamma_{i1} + s_{i2}\gamma_{i2})^{-1} E_i(\phi_{jk}/\phi_{ik}). \tag{2}$$

As is well known, linearity of the stage-1 good implies that the continuation payoff for a country-k resident who leaves the current stage 2 with the portfolio m is

$$w_{ik}(m) = \beta(m_1\phi_{i1}\zeta_{i1} + m_2\phi_{i2}\zeta_{i2}) + A_{ik}, \tag{3}$$

where A_{ik} is a regime-dependent constant. By (3), the rate of return ζ_{ik} cannot exceed $1/\beta$ in any equilibrium. Given this, it is without loss of generality to assume that only consumers who will consume in country k enter stage 2 with currency k. If such a consumer resides in country l and enters stage 2 with m_k units of currency k in state i, then given υ_{ik} as the per-unit price of currency k in the stage-2 country-k market, his continuation payoff is

$$W_{ik}(m_k; l) = \max_{(q, m'_k)} u_{ik}(q) + \beta m'_k \phi_{ik}\zeta_{ik} + A_{il} \tag{4}$$

subject to the CIA constraint

$$q + \upsilon_{ik} m'_k = \upsilon_{ik} m_k. \tag{5}$$

Thus, his trade in the stage-1 market must lead him to enter stage 2 with

$$m_{ik} = \arg\max_{m_k \geq 0} [-m_k\phi_{ik} + W_{ik}(m_k; l)]. \tag{6}$$

Because consumers constitute half the population in each country, stage-1 market clearing requires the average consumer headed to country k to acquire one unit of currency k, i.e.,

$$m_{i1} = m_{i2} = 1 \tag{7}$$

under the flexible-rate regime, and one unit of currency in total, i.e.,

$$m_{i1} + m_{i2} = 2 \tag{8}$$

under the fixed-rate regime.

Let y_{ik} denote the optimal q for the consumer's problem in (4) with $m_k = m_{ik}$. By the envelope condition,

$$\phi_{ik} = \upsilon_{ik} u'_{ik}(y_{ik}).$$

To clear the stage-2 market, a producer in country k must produce y_{ik} so

$$\upsilon_{ik} c'_{ik}(y_{ik}) = \beta \phi_{ik} \zeta_{ik}.$$

When $\beta\zeta_{ik} < 1$, the consumer must spend m_{ik} in the stage-2 market so the CIA constraint (5) becomes $y_{ik} = \upsilon_{ik} m_{ik}$. When $\beta\zeta_{ik} = 1$, the consumer may not spend all m_{ik} but it is without loss of generality to limit attention to equilibria in which he does so. Hence the consumer's optimal condition in the country-k stage-2 market is

$$m_{ik}\phi_{ik} = y_{ik} u'_{ik}(y_{ik}) \tag{9}$$

and the producer's is

$$y_{ik} c'_{ik}(y_{ik}) = \beta m_{ik} \phi_{ik} \zeta_{ik}. \tag{10}$$

Definition 1 *Given a policy γ, a flexible-rate (fixed-rate, resp.) equilibrium is a strictly positive stage-1 price vector $\phi = (\phi_{i1}, \phi_{i2})_{i=1}^{I}$ such that (9) and (10) hold for some allocation y when (7) holds all (i, k) ((8) and $\phi_{i1} = \phi_{i2}$ hold all i, resp.).*

Refer to a positive vector $y = \{(y_{i1}, y_{i2})\}_{i=1}^{I}$ (y_{ik} is stage-2 country-k output in state i) as a stage-2 allocation or simply an *allocation*. Thanks to the linear preference for the stage-1 good, *ex ante* welfare of people in an equilibrium is determined by the allocation of that equilibrium.

An allocation is supported by a regime through a policy-equilibrium pair (γ, ϕ) (or simply supported by a regime) if it is the allocation of an equilibrium ϕ for some policy γ under that regime. The exchange-rate regime is *neutral* if the two regimes support the same set of allocations and *non-neutral* otherwise.

4 Example

Let $I = 2$, the aggregate shock be symmetric, and the policy $\gamma = 1 \in \mathbb{R}^{2I}$ (i.e., $\gamma_{ik} = 1$ all (i, k)). Symmetry in the shock implies symmetry in equilibrium outcomes, i.e., $(a_{21}, a_{22}) = (a_{12}, a_{11})$ for $a = \phi$, m, and y. Let $c(q) = 0.5q^2$ and $u(q) = \ln q + L > 0$ for $q \geq \underline{q} > 0$, where L and $\underline{q}$ are constant and $\underline{q}$ is small.[2] Now the consumer's optimal condition (9) is $m_{ik}\phi_{ik} = \theta_{ik}$ and the producer's optimal condition (10) is $\rho_{ik}y_{ik}^2 = \beta m_{ik} E_i \phi_{jk}$. Set $\theta_{11} + \theta_{12} = 2$, $\rho_{11} + \rho_{12} = 2$, $|\theta_{11} - 1| \in (0, 1 - \beta)$, and $\pi_{11} = 0.5$.

In the flexible-rate equilibrium, the stage-1 market-clearing condition (7) and (9) imply $(\phi_{11}, \phi_{12}) = (\theta_{11}, \theta_{12})$ and $(E_1\phi_{j1}, E_1\phi_{j2}) = (1, 1)$; then by (10),

$$(y_{11}, y_{12}) = (\sqrt{\beta/\rho_{11}}, \sqrt{\beta/\rho_{12}}). \tag{11}$$

In the fixed-rate equilibrium, the stage-1 market-clearing condition (8), $\phi_{11} = \phi_{12}$, and (9) imply $\phi_{11} = \phi_{12} = 1$, $(m_{11}, m_{12}) = (\theta_{11}, \theta_{12})$, and $(E_1\phi_{j1}, E_1\phi_{j2}) = (1, 1)$; then by (10),

$$(y_{11}, y_{12}) = (\sqrt{\beta\theta_{11}/\rho_{11}}, \sqrt{\beta\theta_{12}/\rho_{12}}). \tag{12}$$

As a feature of the utility function u exploited in this example, the shock shifts demands for currencies solely through its effect on the coefficient θ. The flexible-rate regime responds with changes in prices ($(\phi_{11}, \phi_{12}) = (\theta_{11}, \theta_{12})$) while the fixed-rate regime responds with changes in quantities ($(m_{11}, m_{12}) = (\theta_{11}, \theta_{12})$). The future value of one unit of each currency is equal to unity under both regimes so the difference of outputs between (11) and (12) is completely determined by different quantities of currencies generated by the two regimes.

Timing of the shock

The timing of the shock may be chosen so that even with its influence on θ, the shock does not shift the demands for currencies. For this to happen, we can let the two currencies and the linear good be traded in the stage-1 market *before* any shock is realized, and let the two currencies be traded in a *foreign-exchange market* after all shocks are realized but before stage 2 starts. These two markets have the same exchange-rate regime. Now in equilibrium, the price of currency k in the stage-1 mar-

[2] For $0 \leq q < \underline{q}$, one may set $u(q) = 2(q/\underline{q})^{0.5} + \ln \underline{q} + L - 2$, which satisfies $u(0) = 0$, a property used in Proposition 2. All derivations in the example go through if $u(q) = \ln q$ all $q > 0$.

ket is a constant q_k and each person leaves the market with the portfolio $(0.5, 0.5)$. For the flexible-rate regime, let us focus on equilibria in which there is no arbitrage gain for producers between the foreign-exchange market and the coming stage-1 market. Then under each regime, a person can, by trading on the foreign-exchange market, carry the amount of currency k worth of $q = 0.5(q_1 + q_2)$ units of goods in the coming stage-1 market. So by a condition analogous to (9),

$$q = 0.25(1-\lambda)(\theta_{11}+\theta_{21}) + 0.25\lambda(\theta_{12}+\theta_{22}) + 0.5\beta q$$

so $q = 0.5/(1-0.5\beta)$ (recall that λ is the probability for a consumer to be a tourist). Neutrality follows from $\rho_{ik}y_{ik}^2 = \beta q$, a condition analogous to (10). The flexible exchange rate, however, is indeterminate because producers can meet the extra supply or demand of any currency from consumers in the foreign-exchange market when $q_1 + q_2 = 2q$, and q_1 and q_2 are close to each other. This indeterminacy resembles the finding of King et al. [15] when there are no (intrinsic) aggregate shocks.

Real exchange rate

When the exchange-rate regime affects output, it ought to affect the real exchange rate (RER). In this example, I define the RER in state i as

$$RER_i = \frac{\phi_{i1}}{\phi_{i2}} \frac{(1-v_{i1})(1/\phi_{i1}) + v_{i1}(1/y_{i1})}{(1-v_{i2})(1/\phi_{i2}) + v_{i2}(1/y_{i2})}. \tag{13}$$

In (13), $(1-v_{ik})(1/\phi_{ik}) + v_{ik}(1/y_{ik})$ measures the *country-k price level*; $1-v_{ik}$ and v_{ik} are weights for the price of the stage-1 good in the unit of currency k and the price of the stage-2 good produced in country k in the unit of currency k, respectively. The weight v_{ik} is determined by the contribution of the stage-2 domestic output to total country-k output. Because there is no value added in the production of the stage-1 good, net output at stage 1 is zero and, hence, $v_{ik} = 1$. Applying this to (13) and using (9), we have

$$RER_i = u'_{i1}(y_{i1})/u'_{i2}(y_{i2}).$$

Persistence of the shock turns out to be a factor for the RER in the flexible-rate equilibrium. So let $(\pi_{11}, \pi_{12}) = (\mu, 1-\mu)$ and $\mu \geq 0.5$. This generalization does not affect any data for the fixed-rate equilibrium given above. In the flexible-rate equilibrium, $(E_1\phi_{j1}, E_1\phi_{j2}) = (\mu\theta_{11} + (1-\mu)\theta_{12}, \mu\theta_{12} + (1-\mu)\theta_{11})$ and $(y_{11}, y_{12}) = (\sqrt{\beta E_1\phi_{j1}/\rho_{11}}, \sqrt{\beta E_1\phi_{j2}/\rho_{12}})$. Let $\delta_\theta = \theta_{11} - 1$ and $\delta_\rho = \rho_{11} - 1$. By a first-order approximation, the variances of output and the RER in each country are $0.25[(1-2\mu)\delta_\theta+\delta_\rho]^2$ and $[(1+2\mu)\delta_\theta+\delta_\rho]^2$, respectively, in the flexible-rate equilibrium; they are $0.25(\delta_\theta+\delta_\rho)^2$ and $(2\delta_\theta+\delta_\rho)^2$ in the fixed-rate equilibrium. To gauge magnitudes, consider the parameter restriction $\delta_\rho=(\mu-1)\delta_\theta$. With this restriction, the two regimes have the same variance of output, as in Baxter and Stockman [2]. The variance of the RER under the flexible-rate regime is $[3\mu/(1+\mu)]^2$ times that under the fixed-rate regime, which is qualitatively consistent with Mussa [24].

5 General nonneutrality result

Here I establish that given the transition matrix π, nonneutrality holds for almost all shock vectors α in a neighborhood of $1\in\mathbb{R}^{4I}$. To this end, I first show that, when this neighborhood is sufficiently small (so that the implied rates of return satisfy $\zeta_{ik}\le 1/\beta$), there exists a unique flexible-rate equilibrium $\phi(\alpha)$ for the policy $\gamma=1\in\mathbb{R}^{2I}$ (i.e., no lump-sum taxes).

Lemma 1 *Given π and some mild regularity conditions, if α is in a neighborhood of $1\in\mathbb{R}^{4I}$, then there exists a unique flexible-rate equilibrium $\phi(\alpha)$ for the policy $\gamma=1\in\mathbb{R}^{2I}$.*

Proof. The optimal conditions (9) and (10) with $m_{ik}=1$ constitute $4I$ nonlinear equations $F(\phi,y,\alpha)=0$ in unknown (ϕ,y). Here $F=(F_{i1},F_{i2})_{i=1}^I$, $F_{ik}=(F_{ik}^c,F_{ik}^p)$

$$F_{ik}^c(\phi,y,\alpha)=\phi_{ik}-y_{ik}\theta_{ik}u'(y_{ik}), \tag{14}$$

and

$$F_{ik}^p(\phi,y,\alpha)=y_{ik}\rho_{ik}c'(y_{ik})-\beta E_i\phi_{jk}. \tag{15}$$

When $\alpha=1$, $F(\phi,y,\alpha)=0$ has a unique solution (ϕ°,y°), where $y_{ik}^\circ=q^\circ$ and $\phi_{ik}^\circ=q^\circ u'(q^\circ)$ all (i,k), and $\beta u'(q^\circ)=c'(q^\circ)$. As verified in the appendix, the Jacobian matrix of F evaluated at $(\phi^\circ,y^\circ,1)$ is invertible under mild regularity conditions. So by the implicit function theorem, there exists a unique $(\phi(\alpha),y(\alpha))$ for α in a neighborhood N of $1\in\mathbb{R}^4$ such that $F(\phi(\alpha),y(\alpha),\alpha)=0$. By (15), $\phi(\alpha)$ is unique. ■

As it turns out, for generic α, no fixed-rate equilibrium (given any policy) supports the same allocation as the flexible-rate equilibrium $\phi(\alpha)$ in Lemma 1. This fact is the key to the proof of the following result.

Proposition 1 *Given π, the set of α in a neighborhood of $1 \in \mathbb{R}^{4I}$ that permits the two regimes to support the same set of allocations has measure zero.*

Proof. Suppose the two regimes support the same set of allocations. Then some fixed-rate equilibrium for some policy must support the same allocation y as the Lemma-1 flexible-rate equilibrium $\phi(\alpha)$. By (9) and (10),

$$\beta\zeta_{ik}u'_{ik}(y_{ik}) = c'_{ik}(y_{ik}) \tag{16}$$

in each of these two equilibria; that is, stage-2 country-k output in an equilibrium is pinned down by the rate of return of currency k in each state. Because the rates of return of the two currencies are equal in each state in the fixed-rate equilibrium, they must be equal in the equilibrium $\phi(\alpha)$. In $\phi(\alpha)$, $\zeta_{ik} = E_i\phi_{jk}/\phi_{ik}$. Without loss of generality, let $\phi_{12}/\phi_{11} \geq \phi_{j2}/\phi_{j1}$ all j. Then

$$\zeta_{11} - \zeta_{12} = \sum_j \pi_{1j}(\frac{\phi_{j1}}{\phi_{11}} - \frac{\phi_{j2}}{\phi_{12}}) = \sum_j \frac{\pi_{1j}\phi_{j1}}{\phi_{12}}(\frac{\phi_{12}}{\phi_{11}} - \frac{\phi_{j2}}{\phi_{j1}}) = 0 \tag{17}$$

holds only if the exchange rate ϕ_{j2}/ϕ_{j1} is constant in j.[3]

Therefore, to complete the proof, it suffices to show that the set

$$S = \{\alpha : \phi_{i1}(\alpha)\phi_{12}(\alpha) = \phi_{11}(\alpha)\phi_{i2}(\alpha) \text{ all } i\}$$

has measure zero in $\mathbb{R}^{4I}$. To this end, refer to the neighborhood N of $1 \in \mathbb{R}^{4I}$ and $(\phi(\alpha), y(\alpha))$ in the proof of Lemma 1. Let the mapping $\alpha \mapsto \Phi(\alpha) = (\Phi_1(\alpha), \Phi_2(\alpha))$ from N to $\mathbb{R}^{4I}_{++}$ be defined by $\Phi_k(\alpha) = (\phi_{1k}(\alpha), ..., \phi_{Ik}(\alpha), y_{1k}(\alpha), ..., y_{Ik}(\alpha))$. Let the mapping $(\phi, y) \mapsto \Omega(\phi, y)$ from $S' = \{(\phi, y) \in \mathbb{R}^{4I}_{++} : \phi_{i1}\phi_{12} = \phi_{11}\phi_{i2}\}$ to $\mathbb{R}^{I-1}$ be defined by

$$\Omega_i(\phi, y) = \phi_{i1}\phi_{12} - \phi_{11}\phi_{i2}$$

for $2 \leq i \leq I$. Notice that S is the zero set of the composition mapping $\Omega \cdot \Phi$ from N to $\mathbb{R}^{I-1}$. As verified in the appendix, $0 \in \mathbb{R}^{I-1}$ is a regular value of $\Omega \cdot \Phi$. So by the pre-image theorem, the dimension of S is $3I + 1$, as desired. ■

Let me highlight the intuition behind the above argument in the proof of Proposition 1. Neutrality forces the flexible-rate equilibrium $\phi(\alpha)$ to fix the exchange rate for the purpose of maintaining (17), which is an implication of the equilibrium condition (16) (stage-2 output of country k depends on the rate of return of currency

[3] This line of argument is pointed out by Harald Uhlig.

k). But because the equilibrium outcome $(\phi(\alpha), y(\alpha))$ should vary with the physical environment α, fixing the exchange rate limits the freedom of $\phi(\alpha)$ to vary and, hence it can only happen for a measure-zero set of α. The measure-zero set contains any α with $(\theta_{i1}, \rho_{i1}) = (\theta_{i2}, \rho_{i2})$ all i but it may contain other α (e.g., α with $\theta_{11} = \theta_{12}$ in the Section-4 example).

This argument extends readily if the number of state-dependent parameters in α increases, only θ_{ik}s are state-dependent, or only ρ_{ik}s are state-dependent and $q \mapsto qu'(q)$ is strictly monotonic around q°.

Remarkably, Proposition 1 fails if producers supply goods inelastically (e.g., they sell endowed goods as workers in the shopper-worker pairs in the Lucas [20] model). In that case, (10) is not an equilibrium condition so neither is (16). In other words, the flexible-rate equilibrium is not required to have a fixed exchange rate when it supports the same allocation as a fixed-rate equilibrium.

Proposition 1 implies the following inclusion relationship between the two regimes.

Corollary 1 *The set of allocations supported by the flexible-rate regime includes the set of allocations supported by the fixed-rate regime, and given π, the inclusion is strict for generic α in a neighborhood of $1 \in \mathbb{R}^{4I}$.*

Proof. Let y be an allocation supported by the fixed-rate regime through a policy-equilibrium pair (γ, ϕ). Given Proposition 1, it suffices to show that y can be supported by the flexible-rate regime through a policy-equilibrium pair (γ', ϕ'). The idea is to let (γ', ϕ') mimic (γ, ϕ). To be specific, let $m'_{ik} = 1$ and let ϕ' equate the *current* values of m_{ik} and m'_{ik} in the two regimes, i.e.,

$$m'_{ik}\phi'_{ik} = m_{ik}\phi_{ik}. \tag{18}$$

Let γ' equate the *future* values of m_{ik} and m'_{ik}, i.e.,

$$\zeta'_{ik}m'_{ik}\phi'_{ik} = \zeta_{ik}m_{ik}\phi_{ik}.$$

Given (18), this means $\zeta'_{ik} = \zeta_{ik}$ and further by (1),

$$\gamma'_{ik} = (\phi'_{ik}\zeta_{ik})^{-1}E_i\phi'_{jk}. \tag{19}$$

It is straightforward to verify that (18) and (19) are necessary and sufficient for (9) and (10) to hold when $(m'_{ik}, \phi'_{ik}, \zeta'_{ik})$ is substituted for $(m_{ik}, \phi_{ik}, \zeta_{ik})$. So y is indeed supported by the flexible-rate regime through (γ', ϕ'). This completes the proof. ■

6 Taxation constraint and optimal regime

Corollary 1 shows that the flexible-rate regime can support any allocation supported by the fixed-rate regime (and more for generic fundamentals). This might suggest that the flexible-rate regime is at least weakly superior. But is it really the case?

Mimicking a fixed-rate policy-equilibrium pair, as in the proof of Corollary 1, the flexible-rate regime may need to raise positive taxes in some states. When taxes are positive in equilibrium, payment enforcement requires sufficient coercive power. This power is freely assumed in the standard lump-sum tax setup. In the model, however, the severest punishment for tax rebellion is permanent exclusion from future market activities after stage 1 closes. Internal consistency thus renders coercive power endogenous, requiring equilibria to satisfy a taxation constraint.

To describe the taxation constraint, let an allocation y be supported by a regime through a policy-equilibrium pair (γ, ϕ). Let i be the *current* state and h be the *previous* state. If a country-k resident pays the current taxes τ_{hik}, then his continuation payoff is

$$-c_{ik}(y_{ik}) + \beta m_{ik}\phi_{ik}\zeta_{ik} + A_{ik} = -c_{ik}(y_{ik}) + y_{ik}c'_{ik}(y_{ik}) \tag{20}$$

as a producer and

$$u_{il}(y_{il}) - m_{il}\phi_{il} + A_{ik} = u_{il}(y_{il}) - y_{il}u'_{il}(y_{il}) \tag{21}$$

as a consumer to consume in country l, where A_{ik} is given in (3), the equality in (20) uses (10), and the equality in (21) uses (9). Let

$$U_{ik} = \min\{-c_{ik}(y_{ik}) + y_{ik}c'_{ik}(y_{ik}), \min_{l\in\{1,2\}}\{u_{il}(y_{il}) - y_{il}u'_{il}(y_{il})\}\}.$$

Then for all country-k residents to comply, it is necessary that

$$\tau_{hik} \leq U_{ik} + A_{ik}. \tag{22}$$

Let T_{hi} denote the total tax revenue (when i is the current state and h is the previous state) so $T_{hi} = \sum_k \tau_{hik}$. By (22), the total tax revenue for the pair (γ, ϕ) is constrained by *the taxation constraint*

$$T_{hi} \leq \sum_k U_{ik} + \sum_k A_{ik}. \tag{23}$$

To examine the bite of the taxation constraint (23) across regimes, we first compute the tax revenue required to support a given allocation y for each regime.

Lemma 2 *Let y be an allocation supported by the fixed-rate regime through a policy-equilibrium pair (γ, ϕ) and by the flexible-rate regime through a pair (γ', ϕ'). Let $\varphi_{jk} = y_{jk}u'_{jk}(y_{jk})$ and $\varphi_j = \sum_k \varphi_{jk}$. Let ζ_h denote the common value of ζ_{h1} and ζ_{h2} in the fixed-rate equilibrium ϕ. Then the fixed-rate pair (γ, ϕ) and the flexible-rate pair (γ', ϕ') require the tax revenues*

$$T_{hi} = 0.5[\zeta_h \varphi_i \varphi_h (E_h \varphi_j)^{-1} - \varphi_i] \text{ and } T'_{hi} = 0.5[\zeta_h \sum_k \varphi_{ik}\varphi_{hk}(E_h\varphi_{jk})^{-1} - \varphi_i], \quad (24)$$

respectively, to withdraw currencies.

The proof of Lemma 2 is in the appendix. By (9), φ_{jk} in Lemma 2 is the current value of all currency k in state j measured in the stage-1 goods unit for both regimes, i.e.,

$$\varphi_{jk} = m_{jk}\phi_{jk} = m'_{jk}\phi'_{jk}. \quad (25)$$

Using $E_h a_i = E_h a_j$ and (24), one gets

$$E_h T'_{hi} = E_h T_{hi} = 0.5(\zeta_h \varphi_h - E_h \varphi_i); \quad (26)$$

that is, given the previous state h, *before* the current state i is revealed, the two regimes require the same expected tax revenue. Of course, as one can tell from (24), *after* i is realized, the required tax revenue depends on h and on the regime.

Now let us focus on the values of the shock vector α such that there exists a unique positive y^*_{ik} satisfying $u'_{ik}(y^*_{ik}) = c'_{ik}(y^*_{ik})$ all (i, k). Refer to $y^* = \{(y^*_{i1}, y^*_{i2})\}_{i=1}^I$ as the *efficient* allocation (as it maximizes *ex ante* welfare). Referring to (16), one sees that when $y = y^*$, policies γ and γ' in Lemma 2 must stabilize the rate of return of each currency in any state at $1/\beta$ (i.e., $\zeta_{ik} = 1/\beta$ all (i, k)). This, of course, is the Friedman rule.

Lemma 3 *Let $g_{hi} = \varphi_i\varphi_h(E_h\varphi_j)^{-1}$ and $f_{hi} = \sum_k \varphi_{ik}\varphi_{hk}(E_h\varphi_{jk})^{-1}$. Let $\pi_{ij}(t)$ be the t-step transition probability from state i to state j and*

$$V_i(\beta) = 2\beta \sum_k U_{ik} + \beta \sum_{t \geq 1} \sum_j \beta^t \pi_{ij}(t) \sum_k [u_{jk}(y_{jk}) - c_{jk}(y_{jk})].$$

When $y = y^*$, the taxation constraints for the fixed-rate pair (γ, ϕ) and the flexible-rate pair (γ', ϕ') in Lemma 2 can be written as $g_{hi} \leq V_i(\beta)$ and $f_{hi} \leq V_i(\beta)$, respectively.

The proof of Lemma 3 is in the appendix. In Lemma 3, g_{hi} and f_{hi} are the tax revenues required to implement the Friedman policies γ (fixed-rate) and γ' (flexible-rate), respectively, while $V_i(\beta)$ is the continuation payoff. Thus, the two regimes rely on the same continuation payoff to support y^*, but face different history-dependent taxation requirements.

Because g_{hi} and f_{hi} are determined solely by y^* and $V_i(\beta)$ depends only on y^* and β, the tax constraint for either regime is not binding when people are sufficiently patient. The question is which regime supports y^* for a wider range of discount factors.

Lemma 4 *Define $\beta_{flex}(h,i)$ by $f_{hi} = V_i(\beta_{flex}(h,i))$ and $\beta_{fix}(h,i)$ by $g_{hi} = V_i(\beta_{fix}(h,i))$. Let $\beta_{flex} = \max_{(h,i)} \beta_{flex}(h,i)$ and $\beta_{fix} = \max_{(h,i)} \beta_{fix}(h,i)$. In the presence of the tax constraint, y^* is supported by the flexible-rate regime iff $\beta \geq \beta_{flex}$ and by the fixed-rate regime iff $\beta \geq \beta_{fix}$.*

Proof. Note that $V_i(0) = 0$, $V_i(1) = \infty$, and $V_i'(\beta) > 0$. Because f_{hi} and g_{hi} do not depend on β, the two cutoff values of the discount factor $\beta_{flex}(h,i)$ and $\beta_{fix}(h,i)$ are well defined. Because $V_i'(\beta) > 0$, the taxation constraint $g_{hi} \leq V_i(\beta)$ (fixed rate) can hold in the state pair (h,i) iff $\beta \geq \beta_{fix}(h,i)$, and $f_{hi} \leq V_i(\beta)$ (flexible rate) can hold iff $\beta \geq \beta_{flex}(h,i)$. Because tax compliance must hold in all state pairs, y^* is supported by the flexible-rate regime iff $\beta \geq \beta_{flex}$ and by the fixed-rate regime iff $\beta \geq \beta_{fix}$. ■

When the two cutoff values β_{flex} and β_{fix} in Lemma 4 differ, one regime dominates the other over a range of values of β—the fixed-rate regime dominates if $\beta_{flex} > \beta_{fix}$.

Proposition 2 *Suppose (i) given π, α is outside a measure-zero set in $\mathbb{R}^{4I}$, and (ii.a) α implies a sufficiently small cross-state variation in $\sum_k y_{ik}^* u_{ik}'(y_{ik}^*)$ or (ii.b) (π, α) is sufficiently close to a symmetric (π', α') with π' representing an i.i.d. process. Then $\beta_{flex} > \beta_{fix}$. Outside these conditions, neither regime is unambiguously superior.*

Proof. Consider any shock vector α implying $\varphi_{i1} E_h \varphi_{j2} \neq \varphi_{i2} E_h \varphi_{j1}$ all (h,i), which, as verified in the appendix, is a generic property given π. Because

$$f_{hi} - g_{hi} = L_{hi}\left(\frac{\varphi_{h1}}{\varphi_{h2}} - \frac{E_h\varphi_{j1}}{E_h\varphi_{j2}}\right)\left(\frac{\varphi_{i1}}{\varphi_{i2}} - \frac{E_h\varphi_{j1}}{E_h\varphi_{j2}}\right) \tag{27}$$

for some $L_{hi} > 0$, $f_{hi} \neq g_{hi}$ all (h,i). To continue, suppose either (a) φ_i is constant in i or (b) the aggregate shock is symmetric and π represents an i.i.d. process. We claim that given any current state i, for any previous state h, there is a previous state h'

satisfying $f_{h'i} > g_{hi}$ (the flexible-rate regime experiences a larger tax spike than the fixed-rate regime in state i). Because i is arbitrary, the claim implies $\beta_{flex} > \beta_{fix}$.

To verify the claim, first let (a) hold and so by definition (see Lemma 3), $g_{hi} = g_{ii}$. But by (27), $f_{ii} > g_{ii}$; that is, $h' = i$. Next let (b) hold and so $g_{\sigma(h)i} = g_{hi}$ (see Section 2 for σ), $(\varphi_{h1}, \varphi_{h2}) = (\varphi_{\sigma(h)2}, \varphi_{\sigma(h)1})$, and the value of $E_h\varphi_{jk}$ does not depend on (h, k). Then by (27), $\max\{f_{hi}, f_{\sigma(h)i}\} > g_{hi}$; that is, $h' = h$ or $\sigma(h)$.

Because $\beta_{flex} > \beta_{fix}$ holds for any α satisfying condition (a) and any (π, α) satisfying condition (b), continuity implies that the inequality persists for $\tilde{\alpha}$ in a neighborhood of such α (corresponding to condition (ii.a) in the proposition) and for $(\tilde{\pi}, \tilde{\alpha})$ in a neighborhood of such (π, α) (corresponding to condition (ii.b)).

An example in the appendix illustrates the opposite case, $\beta_{flex} < \beta_{fix}$, when neither condition (ii.a) nor (ii.b) holds. ■

Conditions (a) and (b) in the proof of Proposition 2 are intuitive. They ensure that worldwide consumption–production conditions remain stable across all states (condition (a)) or across each pair of symmetric states (condition (b)). Under this stability, fixing exchange rates proves beneficial. By integrating the separated currency values φ_{1i} and φ_{2i} of the two currencies into the unified value φ_i, the fixed-rate regime ensures that for each current state, the required tax revenue (or the stabilization cost) is constant over all previous states or over each pair of symmetric previous states. In contrast, under the flexible-rate regime, each separated currency value φ_{ik} fluctuates with the country-specific consumption-production conditions, resulting in variation in the required tax revenue. In short, when worldwide fundamentals are stable, allowing each currency's value to fluctuate independently is counterproductive as it entails larger tax spikes.

I do not find conditions as easily described and verified as conditions (a) and (b) that ensure reversal of regime superiority (i.e., $\beta_{flex} < \beta_{fix}$). An obvious direction is that the fixed-rate regime entails a larger tax spike in the history that the flexible-rate regime requires the highest discount factor to support the efficient allocation (i.e., $f_{hi} < g_{hi}$ for some state pair (h, i) that attains β_{flex}). This is the case in the appendix example, where the shocks create strong asymmetries across countries.

7 Concluding remarks

This paper offers a monetary perspective on exchange-rate regimes in an environment with aggregate shocks and elastic production. The flexible-rate regime generically yields different allocations than the fixed-rate regime (Proposition 1) and can mimic any fixed-rate equilibrium when taxation is unconstrained (Corollary 1). Once taxation is subject to an endogenous participation constraint, neither regime unambiguously supports the efficient allocation over a larger parameter space (Proposition 2).

A key takeaway from Proposition 2 is that monetary flexibility need not always be a virtue. Flexible exchange rates allow currency-specific returns to respond to country-specific conditions, while fixed exchange rates—by integrating currency values—respond only to worldwide conditions. When worldwide fundamentals are sufficiently stable, such integration entails smaller tax spikes by eliminating unnecessary relative-price adjustments.

Two natural questions arise. First, does the finding depend on the country-specific cash-in-advance constraints? It is well known that those constraints yield domestic-market outcomes that are not in the core for all participants in each country. Second, what is the optimal exchange-rate system? Addressing this question requires developing a more general mechanism for managing relative currency values. Both questions lie beyond the scope of the present paper and naturally motivate subsequent research.

Appendix

Completion of the proof of Lemma 1

To show that the Jacobian matrix of F with respect to (ϕ, y) evaluated at $(\phi, y, \alpha) = (\phi^\circ, y^\circ, 1)$ is invertible, it suffices to show that the Jacobian matrix $\partial F_{\phi yk}$ of $(F_{1k}, ..., F_{Ik})$ with respect to (ϕ, y) evaluated at $(\phi, y, \alpha) = (\phi^\circ, y^\circ, 1)$ is invertible for each k. Notice that $\partial F^c_{ik}/\partial\phi_{ik} = 1$, $\partial F^p_{ik}/\partial\phi_{jk} = -\beta\pi_{ij}$ (all j), $\partial F^c_{ik}/\partial y_{ik} = -\theta_{ik}[u'(y_{ik}) + y_{ik}u''(y_{ik})]$, and $\partial F^p_{ik}/\partial y_{ik} = \rho_{ik}[c'(y_{ik}) + y_{ik}c''(y_{ik})]$; and $\partial F^c_{ik}/\partial\phi_{jk}$, $\partial F^c_{ik}/\partial y_{jk}$, and $\partial F^p_{ik}/\partial y_{jk}$ vanish if $j \neq i$. Hence,

$$\partial F_{\phi yk} = \begin{bmatrix} \mathbf{I} & \vdots & -D_1\mathbf{I} \\ \cdots & \cdots & \cdots \\ -\beta\mathbf{\Pi} & \vdots & D_0\mathbf{I} \end{bmatrix}, \tag{28}$$

where $D_0 = c'(q^\circ) + q^\circ c''(q^\circ)$, $D_1 = u'(q^\circ) + q^\circ u''(q^\circ)$, $\mathbf{I}$ is the $I \times I$ identity matrix, and $\mathbf{\Pi} = (\pi_{ij})$.

Now we assume the following regularity condition: $D_0 \neq \beta\pi_{ii}D_1$ all i.

By its structure, $\partial F_{\phi yk}$ is invertible if its ith and $(i + I)$th columns are linearly independent for $1 \leq i \leq I$. This is the case if the matrix formed by the ith and $(i + 1)$th rows of these two columns is invertible. That, in turn, follows from the assumed regularity condition.

Completion proof of Proposition 1

Let $\partial F^{-1}_{\phi yk}$ be the inverse of $\partial F_{\phi yk}$ in (28), and let $\partial F_{\alpha k}$ be the Jacobian matrix of $(F_{1k}, ..., F_{Ik})$ of F_{ik} (see (14) and (15)) with respect to α evaluated at $(\phi, y, \alpha) = (\phi^\circ, y^\circ, 1)$. By the implicit function theorem, the Jacobian matrix $\partial\Phi_k$ of Φ_k evaluated at 1 is $\partial\Phi_k = -\partial F^{-1}_{\phi yk}\partial F_{\alpha k}$. Because

$$\partial F_{\alpha k} = \begin{bmatrix} -q^\circ u'(q^\circ)\mathbf{I} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & q^\circ c'(q^\circ)\mathbf{I} \end{bmatrix},$$

$\partial\Phi_k$ is invertible. It follows that the Jacobian matrix $\partial\Phi$ of Φ evaluated at 1 is invertible. Apparently, the Jacobian matrix $\partial\Omega$ of $\Omega(\phi, y)$ has full rank $I - 1$ ($\partial\Omega_i/\partial\phi_{i1} = \phi_{12}$, $\partial\Omega_i/\partial\phi_{i2} = -\phi_{11}$, $\partial\Omega_i/\partial\phi_{jk}$ vanishes if $j \neq i$, and $\partial\Omega_i/\partial y_{jk}$ vanishes all (i, k)). So the product of $\partial\Omega$ and $\partial\Phi$ has the full rank $I - 1$, implying that

$0 \in \mathbb{R}^{I-1}$ is a regular value of $\Omega \cdot \Phi$.

Proof of Lemma 2

In this proof, we use the relationships between (γ, ϕ) and (γ', ϕ') given in (18) and (19). Let ϕ_i denote the common value of ϕ_{i1} and ϕ_{i2}, and ζ_h the common value of ζ_{i1} and ζ_{i2}. By (25) and (8), $\phi_i = 0.5\varphi_i$.

To compute the fixed-rate tax T_{hi}, let $a_h = s_{h1}\gamma_{h1} + s_{h2}\gamma_{h2}$ so $a_h\zeta_h\phi_h = E_h\phi_j$ (see (2)). In the current stage-1 market, the per-unit price of each currency is ϕ_i/a_h, and $1 - a_h$ units of currencies are withdrawn. So we have

$$T_{hi} = (\phi_i/a_h)(1 - a_h) = \phi_i[\zeta_h\phi_h(E_h\varphi_j)^{-1} - 1] = 0.5[\zeta_h\varphi_i\varphi_h(E_h\varphi_j)^{-1} - \varphi_i], \quad (29)$$

where the second equality uses $a_h\zeta_h\phi_h = E_h\phi_j$, and the third uses $\phi_i = 0.5\varphi_i$.

Now we compute the flexible-rate tax T'_{hi}. In the current stage-1 market, the per-unit price of currency k is ϕ'_{ik}/γ'_{hk}, and $0.5(1-\gamma'_{hk})$ units of currency k are withdrawn. So

$$\begin{aligned} T'_{hi} &= 0.5\sum_k(\phi'_{ik}/\gamma'_{hk})(1 - \gamma'_{hk}) = 0.5\sum_k[\phi'_{ik}\phi'_{hk}\zeta_h(E_h\phi'_{jk})^{-1} - \phi'_{ik}] \\ &= 0.5[\zeta_h\sum_k \phi_i m_{ik}\phi_h m_{hk}(E_h m_{jk}\phi_j)^{-1} - \phi_i m_{ik}] \\ &= 0.5[\zeta_h\sum_k \varphi_{ik}\varphi_{hk}(E_h\varphi_{jk})^{-1} - \varphi_i], \end{aligned}$$

where the second equality uses (19), the third uses (18) (note $m'_{ik} = 1$), and the last uses $\varphi_{ik} = m_{ik}\phi_{ik} = m_{ik}\phi_i$ and $\varphi_i = \sum_k \varphi_{ik}$.

Proof of Lemma 3

Using (20) and (10), we have the following recursive relationship,

$$A_{ik} = 0.5\beta E_i[-2\tau_{ijk} + \Delta_{jk} + 2A_{jk} - \Lambda_{jk}], \quad (30)$$

where $\Lambda_{jk} = \lambda y_{jk}u'_{jk}(y_{jk}) + (1-\lambda)y_{jl}u'_{jl}(y_{jl}) - y_{jk}c'_{jk}(y_{jk})$ and $\Delta_{jk} = (1-\lambda)u_{jk}(y_{jk}) + \lambda u_{jl}(y_{jl}) - c_{jk}(y_{jk})$ with $l \neq k$. Let $A_i = \sum_k A_{ik}$, $\Delta_j = \sum_k \Delta_{jk}$, and $\Lambda_j = \sum_k \Lambda_{jk}$. Using $u'_{jk}(y_{jk}) = c'_{jk}(y_{jk})$, $\Lambda_j = 0$. Then using $T_{ij} = \sum_k \tau_{ijk}$ and $E_iT_{ij} = 0.5\beta^{-1}\varphi_i - 0.5E_i\varphi_j$ (see (26)) , (30) yields

$$A_i = -0.5\varphi_i + 0.5\beta E_i(\Delta_j + \varphi_j + 2A_j). \quad (31)$$

By repeated substitution, (30) leads to

$$A_i = -0.5\varphi_i + 0.5 \sum_{t\geq 1} \sum_j \beta^t \pi_{ij}(t)\Delta_j.$$

Applying this A_i to (23), setting $\zeta_h = \beta^{-1}$ for T_{hi} and T'_{hi} in (24), and using $\Delta_j = \sum_k [u_{jk}(y_{jk}) - c_{jk}(y_{jk})]$, we get $g_{hi} \leq V_i(\beta)$ and $f_{hi} \leq V_i(\beta)$.

Completion of proof of Proposition 2

We first verify that $\varphi_{i1} E_h \varphi_{j2} = \varphi_{i2} E_h \varphi_{j1}$ is not generic. Fix (i, h) and define the mapping $\alpha \mapsto \Gamma(\alpha)$ by $\Gamma(\alpha) = y_{i1} c'_{i1}(y_{i1}) E_h y_{j2} c'_{j2}(y_{j2}) - y_{i2} c'_{i2}(y_{i2}) E_h y_{j1} c'_{j1}(y_{j1})$, where y_{jk} is an implicit function of α determined by $c'_{jk}(y_{jk}) = u'_{jk}(y_{jk})$. When $h \neq i$, $\partial\Gamma/\partial\rho_{h1} = -\varphi_{i2}\pi_{hh}[c'_{h1}(y_{h1}) + y_{h1}c''_{h1}(y_{h1})]\partial y_{h1}/\partial\rho_{h1}$. When $h = i$ and $h' \neq h$, $\partial\Gamma/\partial\rho_{h'1} = -\varphi_{i2}\pi_{hh'}[c'_{h'1}(y_{h'1}) + y_{h'1}c''_{h'1}(y_{h'1})]\partial y_{h'1}/\partial\rho_{h'1}$. Therefore, it follows from $\partial y_{j1}/\partial\rho_{j1} = -c'(y_{j1})[c''_{j1}(y_{j1}) - u''_{j1}(y_{j1})]^{-1}$ that the Jacobian of Γ evaluated at any α has full rank. So the dimension of the zero set of Γ is $4I - 1$.

Now we give an example with $\beta_{fix} > \beta_{flex}$. Let $c(y) = y$ and let u be the same as in the example in Section 4. Let $I = 3$. Let $(\pi_{i1}, \pi_{i2}, \pi_{i3}) = (\mu_1\psi_1, \mu_1\psi_2, \mu_2)$ for $i = 1, 2$ and $(\pi_{i1}, \pi_{i2}, \pi_{i3}) = (\mu_2/2, \mu_2/2, \mu_1)$, where $\psi_1 + \psi_2 = \mu_1 + \mu_2 = 1$. Let (i) $\theta_{31} = \theta_{11} > \theta_{21}$, (ii) $\theta_{32} > \theta_{12} > \theta_{22}$, and (iii) $\theta_{11}/\theta_{12} > E_1\theta_{j1}/E_1\theta_{j2} > \theta_{31}/\theta_{32}$. By (iii) and (27), $g_{13} > f_{13}$. A simple way to ensure that $(1, 3)$ attains β_{flex} is to vary μ_2 and ρ_{ik}. To see how this works, first note that given (i) and (ii), $f_{13} \geq f_{hi}$ for $h, i \in \{1, 2\}$ (note $E_1\theta_{ik} = E_2\theta_{ik}$). Using (i) and (ii) once more, we have $(1, 3) = \text{argmax} f_{hi}$ if $f_{13} > f_{33}$. When μ_2 is close to 0, $\theta_{1k}/E_1\theta_{ik} > \theta_{3k}/E_3\theta_{ik}$ so (i) and (ii) ensure $f_{13} > f_{33}$. Note that linearity of c implies $A_{ik} = 0$. So by adjusting ρ_{ik}, we can further ensure $V_1(\beta) = V_2(\beta) > V_3(\beta)$.

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