Essentiality and Optimal Usage of Money in Contagious Equilibrium

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Abstract

With certain device to send public signals, a contagious equilibrium renders money inessential among patient agents in the Lagos-Wright model. This paper shows that essentiality of money is robust in that as long as money has arbitrarily small intrinsic value, it is not robust to rely all on public signals (i.e., collective punishment) to discipline sellers and, in particular, only monetary equilibria survive. But money can work better with public signals. A key in the relevant equilibrium is that public signals discipline buyers, which helps sustain a high value of money.

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1 Introduction

The Lagos and Wright (LW) [14] model, a widely used model in recent monetary theory, amends the structure of the Trejos and Wright [16] and Shi [14] (TWS) model by adding a round of centralized trade with a linear good following each round of pairwise meetings. In the TWS and LW models, there are an infinite number of agents and only the buyer and seller can observe the seller's production in each pairwise meeting. So it appears that essentiality of money in the TWS model carries over to the LW model. However, building on the notion of contagious equilibria (see Kandori [9] and Ellison [5]), Aliprantis et al [1] show that there is a non-monetary equilibrium that supports the first-best allocation among patient agents provided that agents can surrender goods to a publicly observed reallocating system in the centralized meeting. In particular, if the seller in a pairwise meeting deviates by not producing for the buyer, then in the succeeding centralized meeting, the buyer and seller surrender an amount of goods to the reallocating system that differs from the amount surrendered by all other agents. This is a public signal which reveals the seller's deviation in the pairwise meeting and, in, turn, triggers global autarky as a threat to deter the deviation.

The main response to Aliprantis et al [1] focuses on setting up the LW model to prevent the reallocating system from sending public signals. Lagos and Wright [15] and Williamson and Wright [20] exclude the reallocating system by only allowing agents to trade goods with money in a Walrasian market in the centralized meeting. Araujo et al [4] modify the centralized meeting and find that money is essential with an infinite number of agents, but not with a finite number of agents. Aliprantis et al [2] preserve the reallocating system but eliminate the public nature of its signals; there are many information-isolated locations that host parallel centralized meetings and agents who meet in a pairwise meeting never visit the same location in the future.

Here I pursue a different approach—an equilibrium outcome with fiat money is not robust if the outcome does not hold when fiat money is replaced by commodity money whose intrinsic value is arbitrarily small but positive (see, e.g., Wallace and Zhu [19]). Specifically, I study the LW model with money *and* the reallocating system sending public signals. Money is part of the physical environment in the model because the presence of durable objects that can be used as money is part of reality and, hence, it is reasonable to follow Wallace [18, p 850] to examine essentiality of money by comparing the monetary-equilibrium outcome and the nonmonetary-equilibrium outcome (holdings of money are ignored). I show two results. First, essentiality of money is robust (according to the above notion of robustness). Second, at least for a class of familiar preferences, money works better with public signals.

Essentiality of money is robust because it is not robust to rely *all* on public signals to *discipline the seller's behavior*. Relying all on public signals on the equilibrium path implies that in the punishment phase, commodity money only has the fundamental value and, hence, a gross real rate of return which is the inverse of the discount factor. If collective punishment is global autarky as in Aliprantis et al [1], then there is no output of goods and, hence, no money-for-goods trade in any pairwise meeting; but because money is valuable, no trade is not in the pairwise core. Although collective punishment need not be global autarky, the high rate of return of money makes it inefficient for any buyer and seller to stay away from some money-for-goods trade that increases output when output is low in their meeting provided that there is no *further* collective punishment to prevent such trade. Because global autarky itself is the limit to collective punishment, commodity money effectively eliminates all nonmonetary equilibria.

Money works better with public signals in part because public signals *discipline the buyer's behavior*. As found by Hu et al [8], if the buyer's share of surplus from trade in a pairwise meeting depends on his payments and drops at the equilibrium level of payments, then agents are willing to acquire money in the centralized meeting by a high price, resulting in a high equilibrium value of money. Different from Hu et al [8], I assume that the buyer makes a take-it-or-leave-it offer in the meeting. Thus dependence of the buyer's share of surplus on his payments can only be maintained if public signals can discipline the buyer with below-equilibrium money holdings to not increase his share in his offer. What enables public signals to do so is an ordinary monetary equilibrium, which has a low equilibrium value of money and which is the equilibrium in the continuation subgame following revelation of the buyer's misbehavior.

2 The model

Time is discrete, dated as $t \geq 1$. There is a continuum set I of infinitely-lived agents. Let $i \in I$ denote a generic agent's identity index. Agent i is always anonymous in that his index is not observed by any $i' \in I \backslash \{i\}$. There is a durable and intrinsically useless good, called money, whose stock is fixed and each agent holds one unit of money at the start of period 1. Each period t has two stages and each stage has a produced good that perishes at the end of the stage.

Stage 1. Each agent has a linear preference on the stage good, i.e., the disutility from producing q units of the good and the utility from consuming q are both equal to q. The individual production is bounded above by an arbitrarily large number. In this stage agents visit a reallocating system and a trading post sequentially.

The reallocating system. Here agent i surrenders s_{it} units of goods simultaneously with all other agents and then receives $\int_{j \in I} s_{jt}$ units of goods from the system. The public observe and only observe the set of surrenderings $S_t = \{s : s = s_{it} \text{ for some } i\}$ (i.e., if $s \in S_t$ then everyone knows that at least one agent surrenders s in period t but the identity of that agent is private information).

The trading post. Here agent i submits an order (q_{it}, p_{it}) with $q_{it} p_{it} = 0$ to the post simultaneously with all other agents: $q_{it} > 0$ means that i is willing to sell q_{it} units of goods and $p_{it} > 0$ means that i is willing to buy goods with p_{it} units of money. The order (q_{it}, p_{it}) is not observed by any agent $i' \neq i$ but the aggregate $(Q_t, P_t) = (\int_{j \in I} q_{jt}, \int_{j \in I} p_{jt})$ is publicly observed. If $Q_t P_t > 0$, then agent i receives $q_{it} P_t / Q_t$ units of money from the post by selling q_{it} units of goods to the post, or receives $p_{it} Q_t / P_t$ units of goods from the post by paying p_{it} units of money to the post. If $Q_t P_t = 0$, then there is no exchange between i and the post. To rule out the self-fulfilling no-trade equilibrium in the trading post, I assume that when agent i is indifferent between submitting $p_{it} > 0$ ($q_{it} > 0$, resp.) and $p'_{it} = 0$ ($q'_{it} = 0$, resp.), he submits p_{it} (q_{it}, resp.) .

Stage 2. Agents first are randomly matched in pairs. Next, in each pairwise meeting, with equal probability, one agent becomes a buyer and another becomes a seller. The seller produces the stage good that is only consumed by the buyer. The seller's disutility from producing y is y and the buyer's utility from consuming y is $u(y)$, where $u(0) = 0$, $u' > 0$, and $u'' < 0$. Let $\theta_{it} = 1/0$ if agent i is a seller/buyer. Let -i denote the identity index of i's stage-2 meeting partner; dependence of -i on t is oppressed to simplify notation. In the meeting, the amount of money x_{jt} carried by agent $j \in \{i,\text{-}i\}$ is only observed by j; the buyer proposes to the seller a meeting outcome o_{it}, meaning that the buyer transfers z units of money to the seller and the seller transfers y units of produced goods to the buyer. Let $\varsigma_{i\tau} = 1/0$ if the buyer's proposal is accepted/rejected: if $\varsigma_{i\tau} = 1$, then both agents act according to the buyer's proposal; otherwise, the transfers of money and goods are both 0. The realized transfers of goods and money are only observed by i and -i.

Each agent maximizes expected discount utility with the discount factor $\beta \in (0, 1)$; there is no discounting within a period; and $\beta u'(0) > 1$. This completes the description of the model.

Three remarks of the model are in order. First, if the model drops the reallocating system then it becomes the conventional version of the LW model; if it drops the trading post then it becomes the version of the LW model in Aliprantis et al [1]. Secondly, an individual agent's surrendering of goods in the reallocating system is an infinitessimal in the aggregate unit. One may question whether it is realistic to assume that an infinitessimal can be observed by the public; the assumption is made to show that essentiality of money is robust even when technology can accurately deliver infinitesimal signals to the public.

Lastly, it is realistic to assume that one's money holdings are not observed by his meeting (though many applications of matching models assume the opposite to simplify analysis). This assumption is used in the proof of Proposition 1 to ensure that an agent who acquires a large amount of money is not discriminated by his meeting partner. On a separated note, under this assumption, it may be assumed that what the buyer proposes is a menu of meeting outcomes conditional on the seller's holdings and/or the seller can show part of his holdings to the buyer before the buyer makes the proposal. The results below hold with these alternative setups.

3 Equilibrium

To describe strategies, let $h_{1\tau} = (S_\tau, Q_\tau, P_\tau)$ and $h_1^{t-1} = (h_{1\tau})_{\tau=1}^t$ for $t > 1$; that is, h_1^{t-1} is the history of stage-1 activities up to the start of $t > 1$ observed by the public. Let $h_{1i\tau} = h_{1\tau} \cup (s_{i\tau}, q_{i\tau}, p_{i\tau})$ and $h_{1i}^{t-1} = (h_{1i\tau})_{\tau=1}^t$ for $t > 1$; that is, h_{1i}^{t-1} is the history of stage-1 activities up to the start of $t > 1$ observed by agent i. Let $h_{2i\tau} = (x_{i\tau}, \theta_{i\tau}, o_{i\tau}, \varsigma_{i\tau})$ and $h_{2i}^{t-1} = (h_{2i\tau})_{\tau=1}^t$ for $t > 1$; that is, h_{2i}^{t-1} is the history of stage-2 pairwise meetings up to the start of $t > 1$ observed by agent i. Let $h_i^t = h_{1i}^t \cup h_{2i}^t$. As a convention, let $h_1^0 = h_{1i}^0 = h_{2i}^0 = \emptyset$.

Let $f_i = (f_{1it}^{allc}, f_{1it}^{post}, f_{2it})_{t\geq 1}$ denote a pure strategy of agent i. For each period t, given h_i^{t-1}, f_{1it}^{allc} specifies an amount of goods for agent i to surrender at the reallocating system; given (h_i^{t-1}, s_{it}, S_t), f_{1it}^{post} specifies an order for i to submit at the trading post; and given $(h_i^{t-1}, h_{1it}, x_{it}, \theta_{it})$, f_{2it} specifies a meeting outcome for agent i to propose when $\theta_{it} = 0$ (i is a buyer) and specifies acceptance or rejection for any outcome proposed by -i when $\theta_{it} = 1$ (i is a seller).

I limit attention to equilibria which reveal deviations in pairwise meetings by public signals immediately. Revelation by public signals, of course, means that after observing some deviation in his period-t pairwise meeting, an agent surrenders an amount in the reallocating system in some $\tau > t$ different than the amount that is surrendered when there is no deviation. Immediate revelation means $\tau = t + 1$; I discuss in what sense this is not restrictive in section 6.1.

Formally, refer to a strategy f_i of agent i as a *quasi public strategy* if for each period t, f_{1it}^{allc} depends only on h_1^{t-1} and h_{2it-1}, f_{1it}^{post} only on (h_1^{t-1}, S_t), and f_{2it} only on h_1^t and (x_{it}, θ_{it}). That is, one's history in his period-t pairwise meeting affects the future only through his surrendering in the reallocating system in period $t + 1$.

Definition 1 *A profile of strategies $f = \{f_i\}_{i \in I}$ is an equilibrium if each f_i is a quasi public strategy and f evaluated at any history of the model determines a Nash equilibrium.*

Throughout, let g_t denote a subgame right after agents observe S_t in period t. Given an equilibrium f, let g_{1f} denote the subgame following that agents observe on-the-equilibrium-path S_1. Let the following be objects specified by f at period $t \geq \tau$ after enter subgame g_τ: $\phi_t(g_\tau, f)$ is the trading-post price of money, $x_{it}(g_\tau, f)$ is money holdings held by agent i at the end of stage 1, and $y_{it}(g_\tau, f, j)$ is output in the meeting between i and -i when $j \in \{i\text{,-}i\}$ is the buyer.

Definition 2 *An equilibrium f is a non-monetary equilibrium if $y_{it}(g_{1f}, f, j)$ does not depend on $(x_{it}, x_{\text{-}it})$ carried by i and -i into their period-t meeting, all $i \in I, t \geq 1$; f is a monetary equilibrium otherwise. A monetary equilibrium is purely monetary if (a) $y_{it}(g_{1f}, f, j)$ only depends on x_{jt} and (b) given x_{jt}, $y_{it}(g_\tau, f, j) = y_{it}(g_{1f}, f, j)$ for any g_τ with $\tau \leq t$, all $i \in I, t \geq 1$.*

Lemma 1 *Let F_{nm} and F_{mm} denote the sets of non-monetary and purely monetary equilibria, respectively.*

(i) If $f \in F_{nm}$ then $\phi_t(g_{1f}, f) = 0$ all t and $y_{it}(g_{1f}, f, j)$ does not depend on i.

(ii) If $f \in F_{mm}$ then $\phi_t(g_{1f}, f) > 0$ all t, $\phi_t(g_\tau, f) = \phi_t(g_{1f}, f)$ for any g_τ with $\tau \leq t$, $x_{it}(g_{1f}, f) = 1$, and given x_{jt}, $y_{it}(g_{1f}, f, j)$ does not depend on i.

Proof. For part (i), let $\phi_t = \phi_t(g_{1f}, f)$ and $y_{it} = y_{it}(g_{1f}, f, j)$. First suppose by contradiction that $\phi_t > 0$ for some t. Because y_{it} does not depend on $(x_{it}, x_{\text{-}it})$, it must be the case that $\phi_{t+1} = \beta^{-1}\phi_t$, leading to an unbounded sequence $\{\phi_\tau\}_{\tau \geq t}$ of the trading-post price of money, which contradicts to the upper bound on the individual stage-1 production. Next suppose by contradiction that $y_{it} > y_{i't}$ for some i and i' when i and i' are buyers at period t. Given -i is willing to accept the proposal to produce y_{it}, -i' must be willing to accept. But i' only requests -i' to produce $y_{i't}$, a contradiction. Part (ii) is standard. ■

I cannot rule out that given $(x_{it}, x_{\text{-}it})$, $y_{it}(g_{1f}, f, j)$ may depend on i when $f \notin F_{mm}$ is a monetary equilibrium. To simplify exposition, I restrict attention to a set F_m of monetary equilibria such that if $f \in F_m$ then $x_{it}(g_{1f}, f) = 1$ all (i, t) and given $(x_{it}, x_{\text{-}it})$, $y_{it}(g_{1f}, f, j)$ does not depend on i. With this restriction, let $y_t(g_{1f}, f)$ denote on-the-equilibrium-path period-t pairwise output in an equilibrium f.

Because of linearity of the stage-1 good, *ex ante* welfare depends only on stage-2 consumption and production. Thus I define the welfare value of f by

$$W(f) = \sum_{t=1}^{\infty} \beta^{t-1} U(y_t(g_{1f}, f)),\ U(y) = 0.5[u(y) - y]. \tag{1}$$

Let $W_a = \sup\{W(f) : f \in F_a\}$, $a \in \{nm, m, mm\}$; as a convention, set $W_a = -1$ if F_a is empty. Notice that

$$W_a \leq W^* \equiv (1-\beta)^{-1} U(y^*),\ y^* = \arg\max_{y \geq 0} U(y). \tag{2}$$

Definition 3 *Money is essential if* $W_m > W_{nm}$.

Aliprantis et al [1] show that when β exceeds a cutoff value, some $f \in F_{nm}$ has $y_t(g_{1f}, f)$ equal to the first-best pairwise output y^* all t. In that equilibrium, if the seller deviates in the period-t meeting between i and -i, then i and -i surrender 0 in the reallocating system at $t+1$ when all other agents surrender some $s > 0$, turning all future pairwise output to 0.

Lemma 2 *When* β *exceeds some cutoff value, money is not essential in the sense of Definition 3.*

4 Commodity-monetary refinement and essentiality of money

The model in section 2, as all standard monetary models, assumes that money is fiat money and, in particular, is intrinsically useless. One may view this assumption as an approximation to the more realistic situation that money bears some small intrinsical value. For example, paper money may be a piece of fine artwork and even in the digital form, money may find its use in a virtual world. Hence one shall regard an equilibrium outcome obtained from a model with fiat money robust only if the outcome is valid as a limiting outcome—a limit of outcomes in the same model when money yields some direct utility and the direct utility approaches zero. In other words, an equilibrium outcome for fiat money is robust only if it survives certain commodity-money refinement.

To apply a commodity-money refinement, I perturb the model in section 2 by letting x units of money yield a direct period utility ϵx at the start of each period, where $\epsilon > 0$ can be arbitrarily small. From now now, I refer to such a model as *the perturbed model.*

The categorization of equilibria in Definition 2 carries over to the perturbed model without any change; moreover, Lemma 1 can be adapted as follows.

Lemma 3 *For the perturbed model, Lemma 1 applies except that now if $f \in F_{nm}$ then $\phi_t(g_{1f}, f) = \alpha(\epsilon) \equiv \beta(1-\beta)^{-1}\epsilon$ all t and if $f \in F_{mm}$ then $\phi_t(g_{1f}, f) > \alpha(\epsilon)$ all t.*

Proof. For $f \in F_{nm}$, refer to the proof of Lemma 1 and independence of y_{it} on i follows from the argument there. By that independence, $\phi_{t+1} + \epsilon = \beta^{-1}\phi_t$. If $\phi_t > \alpha(\epsilon)$, then $\phi_{t+1} - \phi_t = (\beta^{-1} - 1)\phi_t - \epsilon > (\beta^{-1} - 1)\alpha(\epsilon) - \epsilon = 0$, leading to a strictly increasing sequence $\{\phi_\tau\}_{\tau \geq t}$; but because $(\beta^{-1} - 1)\phi_t - \epsilon$ is strictly increasing in t, $\{\phi_\tau\}_{\tau \geq t}$ is unbounded. Results for $f \in F_{mm}$ are standard. ■

By imposing the same restriction to the set F_m of monetary equilibria as in section 2, I now refine the notion in Definition 3 as follows.

Definition 4 *Money is essential if $W_m - W_{nm}$ in the perturbed model is positively bounded below as $\epsilon \to 0$.*

Essentiality of money in the sense of Definition 4 is built on nonexistence of any nonmonetary equilibrium in the perturbed model.

Proposition 1 *For the perturbed model, F_{nm} is empty.*

Proof. See section 6.1. ■

It helps spell out the basic idea of the proof of Proposition 1 here. Suppose there exists a non-monetary equilibrium f. Because money has an intrinsic value, output in pairwise meetings cannot be zero on the equilibrium path of g_{1f}. Because f is a non-monetary equilibrium, positive pairwise output in g_{1f} can only be sustained through collective punishment triggered by public signals. That is, if the seller in a pairwise meeting in period t deviates to not produce, then everyone's continuation value v (conditional on that one holds one unit of money) at the start of the subgame g following the revelation of the deviation in period $t+1$ by public signals falls below his continuation value along the equilibrium path of g_{1f} following no revelation of any deviation.

In the subgame g, money remains to only have the fundamental value—otherwise, anyone on the equilibrium path of g_{1f} can earn arbitrarily large amount of profits by buying money from the trading post in period t and sending a revealing signal in period $t+1$ that leads to g. The main argument in the proof is that some deviation in g must drive the continuation value down below v. This substantially uses the fact that money remains to only have the fundamental value, and this fact is critical because money has an intrinsic value. Indeed, if money has no intrinsic value, then its fundamental value is zero and it can be arranged that in g, output in pairwise meetings is sufficiently low in the first few periods (to ensure that v is lower than the continuation value along the equilibrium path of g_{1f}) and output returns to the path of g_{1f}; with this arrangement, there is no need to have a continuation value lower than v.

But when money has an intrinsic value and its equilibrium value is the fundamental value, any agent has incentive to carry a sufficient amount of money into pairwise meetings if output in meetings is supposed to be low—if I become the buyer in the meeting, I can induce high output by giving you large payments. What if I become a seller? The rate of return of money compensates discounting and because I can hide my money holdings, I can ensure that you do not discriminate me in your proposal. Therefore, to prevent me from acquiring a large amount of money in the trading post and then trading money for high output with you (when I am the buyer), it is necessary to reveal our pairwise trade by public signals, signals that lead to a continuation value w lower than v.

Repeating this argument, the nonmonetary equilibrium must admit a decreasing sequence of continuation values. But when the continuation value moves down, the potential gain from the good-for-money trade moves up, calling for an increase in the degree of collective punishment, which leads to a contradiction because punishment cannot go beyond global autarky.

As is well known, the set F_{mm} of purely monetary equilibria is nonempty and, in particular, has a unique stationary equilibrium (see, e.g., Gu and Wright [7]).

Lemma 4 *For the perturbed model, F_{mm} contains a unique stationary equilibrium whose constant trading-post price of money ϕ and pairwise output y are determined by $0.5y[u'(y)+1]=\phi$ and $y=\beta(\phi+\epsilon)$.*

Because pairwise output y in the Lemma-4 stationary equilibrium is bounded away from 0 as $\epsilon\to 0$, we reach the following.

Corollary 1 *Money is essential in the sense of Definition 4.*

5 Coessentiality of money and public signals

Essentiality of money does not rule out usefulness of public signals. Usefulness of public signals can be addressed in the section-2 model and in the perturbed model. To simplify exposition, I work with the section-2 fiat-money model. A precise sense of usefulness of public signals is the following.

Definition 5 *Provided that money is essential, money and public signals are coessential if $W_m > W_{mm}$.*

The following result for $f\in F_{mm}$ is well known (see, e.g., Gu and Wright [7]).

Lemma 5 *Any $f\in F_{mm}$ satisfies $\phi_t=0.5\beta y_t[u'(y_t)+1]$ and $y_t=\beta\phi_{t+1}$, where $y_t=y_t(g_{1f},f)$ and $\phi_t=\phi_t(g_{1f},f)$; moreover, F_{mm} contains a unique stationary equilibrium f° with $y_t=y^\circ$ and $\phi_t=\phi^\circ$ all t.*

For coessentiality, first refer to the Lemma-5 stationary equilibrium f°. Note $u'(y^\circ)=2/\beta-1$ so $U'(y^\circ)=1/\beta-1$. If

$$2[\frac{U(y^*)}{1-\beta}-\frac{U(y^\circ)}{1-\beta}]>\frac{y^*}{\beta}-\frac{y^\circ}{\beta}, \tag{3}$$

then set $\bar{y} = y^*$ and pick s satisfying

$$2[\frac{U(\bar{y})}{1-\beta} - \frac{U(y^\circ)}{1-\beta} - s] = \frac{\bar{y}}{\beta} - \frac{y^\circ}{\beta}; \tag{4}$$

otherwise, pick a small $s > 0$ such that there exists a unique $\bar{y} \in (y^\circ, y^*)$ satisfying (4). Let $\bar{\phi}$ be determined by

$$\frac{U(\bar{y})}{1-\beta} - \frac{U(y^\circ)}{1-\beta} - s = \bar{\phi} - \phi^\circ. \tag{5}$$

Proposition 2 *There exists $\bar{f} \in F_m$ such that $\phi_t(g_{1f}, \bar{f}) = \bar{\phi}$, $y_t(g_{1f}, \bar{f}) = \bar{y}$, and $S_t = \{s\}$ on the equilibrium path all t.*

Proof. See section 6.2. ■

In the Proposition-2 equilibrium $\bar{f}$, the buyer in a pairwise meeting proposes to trade one unit of money with $\bar{y}$ units of goods on the equilibrium path. By $y^\circ = \beta\phi^\circ$, (4) and (5) imply

$$\frac{U(\bar{y})}{1-\beta} - \frac{U(y^\circ)}{1-\beta} - s = \frac{\bar{y}}{\beta} - \bar{\phi}. \tag{6}$$

Because $\bar{y} > y^\circ$, (4) and (6) imply $\bar{y} > \beta\bar{\phi}$; that is, monetary rewards only compensate part of the seller's production disutility. If the seller deviates to reject the proposal, then agents play the equilibrium f° after the buyer and seller surrender 0 (while all other agents surrender s, which is positive) at the reallocating system in the next period; by (6), the seller cannot benefit from the deviation.

Two telling observations help explain why $\bar{f}$ mixes monetary rewards and collective punishment to incentive sellers to produce $\bar{y}$. First, although money need not cover all the seller's disutility in the pairwise meeting, the trading-post price of money must exceed ϕ° for collective punishment (i.e., playing f°) to be effective. For, otherwise, the disutility not compensated by money exceeds $\bar{y} - \beta\phi^\circ = \bar{y} - y^\circ$; but, then, by $U'(y^\circ) = 1/\beta - 1$ and strict concavity of U,

$$\frac{U(\bar{y}) - U(y^\circ)}{1-\beta} < \frac{\bar{y} - y^\circ}{\beta}, \tag{7}$$

meaning that collective punishment is not sufficient to deter the seller from deviating. Second, the trading-post price ϕ of money cannot cover all the seller's disutility in the pairwise meeting. For, otherwise, $\bar{y} \leq \beta\phi$ and, hence, $\beta(\phi - \phi^\circ) \geq \bar{y} - y^\circ$; but, then, (7) (which follows from $U'(y^\circ) = 1/\beta - 1$) implies

$$\frac{U(\bar{y}) - U(y^\circ)}{1-\beta} < \phi - \phi^\circ,$$

meaning that an agent who enters the current period with the zero units of money finds it beneficial to trigger collective punishment. Setting $\phi = \bar{\phi}$ best balances the need to incentivize sellers by monetary rewards to produce $\bar{y}$ in pairwise meetings and the need to incentivize agents to participate in the equilibrium plays under the trading-post price of money $\bar{\phi}$ instead of ϕ°; the latter need, of course, is associated with the individual agent's capacity to turn the price of money from $\bar{\phi}$ to ϕ°.

Now I explain how to induce agents to hold one unit of money when the constant on-the-equilibrium-path trading-post price of money exceeds ϕ°. On the equilibrium path, each seller in a pairwise meeting is supposed to produce $\xi(z)$ units of goods for exchange of z units of money, where

$$\begin{aligned}
&\xi(z) = \bar{y} + \beta\bar{\phi}(z-1),\ z \geq 1, \\
&\xi(z) = \begin{pmatrix} \beta\phi^\circ z,\ z \leq \tilde{z} \\ u^{-1}(\beta\bar{\phi}z),\ z \in (\tilde{z}, 1) \end{pmatrix} \text{ if } \tilde{z} < 1, \\
&\xi(z) = \beta\phi^\circ z,\ z < 1 \text{ if } \tilde{z} \geq 1,
\end{aligned} \tag{8}$$

and

$$\tilde{z} = \max\{z : u(\beta\phi^\circ z) \geq \beta\bar{\phi}z\} \tag{9}$$

One may read (8) as follows. When the value z of the buyer's payments is at least 1, the buyer receives $\bar{y}$ units of goods from paying 1 and the additional $\beta\bar{\phi}(z-1)$ from paying $z-1$. When $z \leq \tilde{z}$ in case $\tilde{z} < 1$ or $z < 1$ in case $\tilde{z} \geq 1$, the buyer makes the offer as if the future trading-post price of money were ϕ°. When $z \in (\tilde{z}, 1)$ in case $\tilde{z} < 1$, the buyer gets the utility $u(\beta\phi^\circ\tilde{z})$ from paying $\tilde{z}$ and the additional utility $\beta\bar{\phi}(z-\tilde{z})$ from paying $z-\tilde{z}$; by (9), $u(\beta\phi^\circ\tilde{z}) = \beta\bar{\phi}\tilde{z}$ so paying z, he receives $u^{-1}(\beta\bar{\phi}z)$ units of goods and his utility is $\beta\bar{\phi}z$.

By design, $\xi(z) < \beta\bar{\phi}z$ for $z \in (0,1)$. (This is clear when $z \leq \tilde{z}$ in case $\tilde{z} < 1$ or $z < 1$ in case $\tilde{z} \geq 1$; when $z \in (\tilde{z}, 1)$ in case $\tilde{z} < 1$, $\bar{y} \leq y^*$ and $\beta\bar{\phi}z < \bar{y}$ imply $\xi(z) < u(\xi(z)) = \beta\bar{\phi}z$.) Therefore, the seller enjoys positive surplus if the buyer happens to carry $x \in (0,1)$ units of money into the meeting. Letting the seller enjoy positive surplus uses the idea of Hu et al [8]. That is, when the buyer's share in surplus split depends on his payments and, in particular, drops at the equilibrium level of payments, the marginal increase in the buyer's utility of consumption due to a marginal increase in money holdings can be high enough to induce agents to acquire money in the stage-1 trading by a price greater than ϕ°.

But why does not the buyer carrying $x \in (0,1)$ deviate from $(\xi(z), z)$ to increase his own surplus share? Hu et al [8] adopt the game form in Zhu [21] that is designed to implement a planned outcome in the meeting-specific pairwise core. That game form allows the buyer to propose a meeting outcome alternative to the planned outcome; because the planned outcome is in the pairwise core and is always feasible for the seller to choose, the buyer does not propose an alternative that increases his share in equilibrium (such alternative cannot be mutually improving).

The game form here is apparently different—when the buyer proposes a meeting outcome alternative to $(\xi(z), z)$ that increases his share, the seller can only accept or reject the proposal. As such, the alternative being not mutually improving (with respect to $(\xi(z), z)$) is not sufficient for the seller to reject it. What deters the buyer is that his deviation triggers playing f° in the continuation subgame. Because the trading-post price of money in the continuation subgame is ϕ° instead of $\bar{\phi}$, the deviation effectively alters surplus itself instead of how the surplus determined by the price $\bar{\phi}$ is split. Indeed, the alternative outcome actually makes both agents worse off under the price ϕ°; this explains why (8) relates $\xi(z)$ to ϕ° for $z < 1$.[1]

Coessentiality holds as long as $W_{mm} = W(f^\circ)$. It can be shown that this is the case for some preferences; e.g., $y \mapsto u'(y)y$ is nondecreasing in y. I suspect that this is the case even for general preferences but I cannot prove it.

Corollary 2 *Money and public signals are coessential whenever preferences imply* $W_{mm} = W(f^\circ)$.

6 Proof of Propositions 1 and 2

6.1 Proposition 1

Assume by contradiction that F_{nm} is nonempty and pick $f \in F_{nm}$. Say a subgame g_{t+1} is *reachable* from a subgame g_t if some actions taken by agents in period t in g_t

[1] If one directly borrows the scheme in Hu et al [8] to set $\xi(z) = u^{-1}(\beta\bar{\phi}z)$ for $z < 1$, then when $u'(0)$ is sufficiently large and z is sufficiently small, the buyer and seller mutually benefit from exchanging z units of money with $\beta\bar{\phi}z$ units of goods.

turn g_{t+1} as the subgame following that agents observe S_{t+1} specified by f. Let G be the set of subgames such that $g_t \in G$ if there exists a sequence of subgames $\{g_\tau\}_{\tau=1}^t$ satisfying that $g_1 = g_{1f}$ and when $t > 1$, g_τ is reachable from $g_{\tau-1}$ all $1 < \tau \leq t$. The rest of the proof is split into a few lemmas. The first lemma establishes that when entering a subgame $g_t \in G$, f specifies agents to trade money in the trading post by its fundamental value $\alpha(\epsilon)$ (see Lemma 3).

Lemma 6 *Let $g_t \in G$. Then $\phi_\tau(g_t, f) = \alpha(\epsilon)$ all $\tau \geq t$.*

Proof. Suppose $\phi_t(g_t, f) = \alpha(\epsilon)$ and we have two intermediate results.

Claim 1. $\phi_\tau(g_t, f) = \alpha(\epsilon)$ all $\tau \geq t$.

Claim 2. $\phi_\tau(g_t^\tau, f) = \alpha(\epsilon)$, where $\tau > t$ is the first $t' > t$ such that $S_{t'}$ observed by agents differs from one that is specified by f for the subgame g_t in period t' (i.e., some agent in t' surrenders some amount of goods that differs from those specified by f for g_t in t') and g_t^τ is the subgame after agents observe such S_τ.

To verify Claim 1 and 2, first notice that if the trading-post prices of money in periods τ and $\tau + 1$ are $\alpha(\epsilon)$, then the gross rate of return of holding money from τ to $\tau + 1$ is $[\epsilon + \alpha(\epsilon)]/\alpha(\epsilon) = 1/\beta$ (one unit of money yields ϵ units of dividend before trading). For Claim 1, it suffices to show that $\phi_{t+1}(g_t, f) = \alpha(\epsilon)$. If $\phi_{t+1}(g_t, f) > \alpha(\epsilon)$, then the gross rate of return of money from t to $t+1$ exceeds $1/\beta$, which is impossible. For Claim 2, suppose otherwise. Then, an agent can gain by acquiring a sufficiently large amount of money at the trading post in $\tau - 1$ and surrendering some amount of goods that differs from those specified by f for g_t in τ and leads to $\phi_\tau(g_t^\tau, f) > \alpha(\epsilon)$. To complete the proof, note by Lemma 3, $\phi_t(g_{1f}, f) = \alpha(\epsilon)$ all t. Because $g_t \in G$, Claim 2 implies $\phi_t(g_t, f) = \alpha(\epsilon)$. ■

When agents enter a subgame $g_t \in G$, money has the fundamental value but we cannot rule out that agents leave the trading post with different amounts of money and pairwise output may vary with the buyer's money holdings. Thus we introduce some objects specified by f at period $\tau \geq t$ for g_t: let $v_\tau(x, g_t)$ be the continuation value of an agent who enters the trading post with x units of money, $\Gamma_\tau(x, g_t)$ be the distribution of money holdings when agents leave the trading post, and $X_\tau(g_t)$ be the support of $\Gamma_t(x, g_t)$; let $(q_\tau(x, g_t), l_\tau(x, g_t))$ be the realized meeting outcome when a buyer holding x amount money meets a seller holding x' at stage 2; and let $\Pi_\tau(x, x', g_t)$ be the distribution of the buyer-seller portfolios (x, x') at stage 2. A recursive link among the above objects is summarized as follows.

Lemma 7 *Let $g_t \in G$. Then $v_\tau(1, g_t) = \int U(q_\tau(x, g_t))d\Pi_\tau(x, x', g_t) + \beta v_{\tau+1}(1, g_t)$ for $\tau \geq t$.*

Proof. By the definitions of $v_\tau(x, g_t)$, $U(q_\tau(x, g_t))$, and $\Pi_\tau(x, x', g_t)$ and the linearity of stage-1 goods, we have

$$\int v_\tau(x, g_t)d\Gamma_\tau(x, g_t) = \int U(q_\tau(x, g_t))d\Pi_\tau(x, x', g_t) + \beta \int v_{\tau+1}(x, g_t)d\Gamma_{\tau+1}(x, g_t);$$

that is, the average continuation value is the sum of the average pairwise utility gain and the discount future average continuation value. Because linearity of stage-1 goods implies $v_\tau(x, g_t) = v_\tau(1, g_t) + (x - 1)\phi_\tau(g_t, f)$, the lemma follows. ■

Let $V = \{v_\tau(1, g_t) : g_t \in G, \tau \geq 1\}$. The next lemma shows that $v \in V$ as a credible equilibrium value must be supported by a threat, namely, a value w that falls below it with some distance.

Lemma 8 *Let $v \in V$. Then there exists $w \in V$ such that $v - w \geq k(v)$, where*

$$k(v) = \begin{pmatrix} 0.5\beta^{-1}y^*, \ v = W^* \\ \beta^{-1}[U(y^*) - U(\hat{y}(v))], \ v < W^* \end{pmatrix},$$
$$\hat{y}(v) = U^{-1}((1 - \beta)v) \leq y^*.$$

Proof. Fix $v \in V$ and pick g_t so that $v_t(1, g_t) = v$. For $\tau \geq t$, let $v_\tau = v_\tau(1, g_t)$, $q_\tau(x) = q_\tau(x, g_t)$, $l_\tau(x) = l_\tau(x, g_t)$, $\Gamma_\tau(x) = \Gamma_\tau(x, g_t)$, $\Pi_\tau(x, x') = \Pi_\tau(x, x', g_t)$, and $X_\tau = X_\tau(g_t)$. Let $\alpha = \alpha(\epsilon)$; let ϵ be sufficiently small so that $\alpha < 0.5y^*$.

First consider $v = W^*$, which implies $v_{t+1} = v$ and $q_t(x) = y^*$ all $x \in X_t$. Consider period t in subgame g_t and fix $x \in X_t$ with $x \leq 1$. Let S_{t+1} be the set of surrenderings in period $t+1$ specified by f if agent -i, the seller in a period-t pairwise meeting deviates to produce 0 for agent i, the buyer who holds x (and there is no deviation in other period-t meetings); let g_{t+1} be the subgame after S_{t+1} is observed; and let $w = v_{t+1}(1, g_{t+1})$. To prevent -i from deviating, it is necessary to have

$$w \leq W^* + \frac{\alpha - y^*}{\beta}. \tag{10}$$

In (10) (and also in (16) and (18) below), the terms of surrenderings do not appear because the amount of surrendering by which an agent reveals a deviation cannot be greater the amount by which the agent does not reveal; the term α appears because

$l_t(x)$ need not be zero and by Lemma 6, $\phi_{t+1}(g_t, f) = \phi_{t+1}(g_{t+1}, f) = \alpha$. It follows from (10) that $w < k(W^*)$.

Next consider $v < W^*$. Let $y_t \in [0, y^*]$ satisfy $U(y_t) = \int U(q_t(x))d\Pi_t(x, x')$; by Lemma 7, $v = U(y_t) + \beta v_{t+1}$. Let $\hat{y} = \hat{y}(v)$ so $v = U(\hat{y}) + \beta v$. It follows that

$$v_{t+1} = v - \frac{U(y_t) - U(\hat{y})}{\beta}. \tag{11}$$

If $y_t = y^*$ then v_{t+1} is the desired w. So consider $y_t < y^*$ and assume by contradiction that

$$v' > v - \frac{U(y^*) - U(\hat{y})}{\beta} \text{ if } v' \in V. \tag{12}$$

Fix $x \in X_t$ with $U(q_t(x)) \leq U(y_t)$. We proceed by assuming $q_t(x) < y^*$; if $q_t(x) > y^*$ then simply set $D = 0$ in the proof below. Consider in the subgame g_t, agent i leaves the period-t trading post with $D/\alpha + x$ units of money, where

$$D = 0.5[u(y^*) + y^*] - 0.5[u(q_t(x)) + q_t(x)]. \tag{13}$$

When i becomes the seller in his period-t pairwise meeting, he behaves as if he holds x in the meeting. When i becomes the buyer, it is feasible for him to propose (y^*, L) with $L = D/\alpha + l_t(x)$ to -i, his meeting partner. By (13), we have

$$(-y^* + \alpha L) - [-q_t(x) + \alpha l_t(x)] = U(y^*) - U(q_t(x)) \tag{14}$$

and

$$[u(y^*) - \alpha L] - [u(q_t(x)) - \alpha l_t(x)] = U(y^*) - U(q_t(x)). \tag{15}$$

Let S_{t+1} be the set of surrenderings in period $t+1$ specified by f if -i accepts (y^*, L) (and there is no deviation in other period-t meetings); let g_{t+1} be the subgame after S_{t+1} is observed; and let $w = v_{t+1}(1, g_{t+1})$. To prevent -i from accepting (y^*, L), it is necessary to have

$$(-y^* + \alpha L) - [-q_t(x) + \alpha l_t(x)] \leq \beta(v_{t+1} - w), \tag{16}$$

Using (14) and the value of v_{t+1} in (11), (16) implies

$$w \leq v - \frac{U(y_t) - U(\hat{y})}{\beta} - \frac{U(y^*) - U(q_t(x))}{\beta}. \tag{17}$$

Because $U(q_t(x)) \leq U(y_t)$, the maximal value of w satisfying (17) is below the lower bound of V in (12). Therefore, -i must accept (y^*, L) if it is proposed. Consequently, leaving the period-t trading post with $D/\alpha + x$ units of money, agent i gets at least

$$\Delta \equiv -D + 0.5[u(y^*) - \alpha L - u(q_t(x)) + \alpha l_t(x) + \beta(w - v_{t+1})] + 0.5D \qquad (18)$$

more expected net payoff than leaving the trading post with x units of money. Using (14) and $D = [L - l_t(x)]\alpha$, we have

$$u(y^*) - \alpha L - u(q_t(x)) + \alpha l_t(x) - D = U(y^*) - U(q_t(x)). \qquad (19)$$

Using the value of v_{t+1} in (11) and the lower bound of V in (12), (18) and (19) imply

$$2\Delta > U(y^*) - U(q_t(x)) + \beta[v - \frac{U(y^*) - U(\hat{y})}{\beta} - v + \frac{U(y_t) - U(\hat{y})}{\beta}] \geq 0.$$

Hence, it is not optimal to leave the trading post with x units of money, i.e., $x \notin X_t$, a contradiction. ■

By Lemma 8, for v^1 to be a credible equilibrium value, V must contain some $v^2 < v^1$; for v^2 to be credible, V must contain some $v^3 < v^2$; and so on. Because v^{n+1} is bounded away from v^n by $k(v^n)$, we reach the following.

Lemma 9 *The set V is empty.*

Proof. Suppose by contradiction that V is not empty. By Lemma 8, there exists a strictly decreasing sequence $\{v^n\} \subset V$ with $v^n - v^{n+1} > k(v^n)$. Because any element in V cannot be negative, the sequence converges. Let $\underline{v} = \lim v^n$. Because $\underline{v} < W^*$, the function $b \mapsto k(b)$ is continuous at $b \neq W^*$, and $k(b) > 0$ for $b \in [0, W^*]$, $v^n - v^{n+1} > k(v^n)$ implies $0 = \lim(v^n - v^{n+1}) \geq \lim k(v^n) = k(\underline{v}) > 0$, a contradiction. So V must be empty. ■

Lemma 9 permits us to conclude that there does not exist any nonmonetary equilibrium in the perturbed model and completes the proof of Proposition 1.

Discussion

Recall that we limit attention to equilibria that reveal deviations in period t pairwise meetings by S_{t+1} through the reallocating system in period $t+1$. Can we rule out a nonmonetary equilibrium admitting no immediate revelation? With no immediate revelation, an equilibrium should consist of a strategy profile f and a belief system μ. Whatever μ is, one shall see that delayed revelation only affects the proof of Lemma 8. Without loss of generality, suppose that deviations in t are revealed in $t+2$. For $v = W^*$, delayed revelation only leads to a smaller value of w. Indeed, ignoring any potential benefit of -i from his further deviation in $t+1$, we can modify (10) as $w \leq W^* + \beta^{-2}(\alpha - y^*)$.

For $v < W^*$, delayed revelation leads to no effective revelation at all in a sense as follows. Delayed revelation implies that agents do not respond to S_{t+1} so f specifies everyone to surrender 0 in $t+1$. Without loss of generality, suppose that f specifies everyone to surrender some $s_{t+2} > 0$ in period $t+1$ if there is no deviation in period t and that f specifies i and -i to surrender 0 in period $t+2$ if they trade y^* units of goods for L units of money in period t when i is the buyer. Now think of i, as the buyer in the period-t meeting, proposes (y^*, L) together with a pair of positive numbers s_i and $s_{\text{-}i}$ and speaks to -i along the following line.

"It is mutually benefiting for us to trade y^* with L if we can ensure that we surrender s_{t+2} in $t+2$. There is a self-insuring mechanism whose unique equilibrium outcome—with a plausible equilibrium concept—is to surrender s_{t+2} in $t+2$ if we trade. I understand that we are playing a game with the whole community. But what really affects our decision now is a game between you and me as we know that the rest of the community just follow f. Therefore, what I mean by the mechanism is a two-person game characterized by $(s_i, s_{\text{-}i})$. Why don't we trade and use this mechanism as the support?"

What is the two-person game? In $t+1$, i/-i chooses to surrender either $s_i/s_{\text{-}i}$ or 0 to the reallocating system; then each $j \in \{i,\text{-}i\}$ chooses an action which represents his activities in the trading post and his pairwise meeting (provided that other agents in $I\backslash\{i,\text{-}i\}$ follow f). In $t+2$, each $j \in \{i,\text{-}i\}$ chooses to surrender s_{t+2} or 0 and the game ends. For each agent, his payoff at the terminal node of the game is the sum his $t+1$ utility/disutility associated with goods and money, his utility associated with surrenderings in $t+1$ and $t+2$, and his continuation value implied by f after S_{t+2} is observed. Because f specifies everyone to surrender 0 in $t+1$ and because s_i and $s_{\text{-}i}$ are picked by i in period t, they feel sure that if $s_j \in S_{t+1}$, $j \in \{i,\text{-}i\}$, it is surrendered by j. If a positive $s \notin \{s_i, s_{\text{-}i}\}$ is in S_{t+1}, the belief of i and -i is that s is incidentally surrendered by someone from $I\backslash\{i,\text{-}i\}$ (recall that i and -i cannot observe who surrenders s).

For this two-person game, what is critical is that any strategy of $j \in \{i,\text{-}i\}$ that specifies j to surrender s_j in $t+1$ and 0 in $t+2$ is strictly dominated by a strategy which specifies j to surrender 0 in $t+1$ and 0 in $t+2$. Applying a notion of forward induction proposed in the literature (e.g., one in Govindan and Wilson [6]), the unique sequential equilibrium of the two-person game that satisfies forward induction is one that agent $\in \{i,\text{-}i\}$ surrenders s_j in $t+1$, takes the action that represents his activities in the trading post and his pairwise meeting in $t+1$ according to f_{jt+1} as if $S_{t+1} = \{0\}$ and i and -i haven't deviated in period t, and surrenders s_{t+2} in t+2. Anticipating this, -i accepts the proposal (y^*, L). Therefore, taking the two-person game as a refining device, we can rule out any non-monetary equilibrium that admits no immediate revelation.

6.2 Proposition 2

To construct $\bar{f}$, we describe actions specified by the strategy $\bar{f}_i = (\bar{f}^{allc}_{1it}, \bar{f}^{post}_{1it}, \bar{f}_{2it})_{t\geq 1}$ fro agent i to take in period t.

(a) The history h^t_1 has $S_\tau = \{s\}$ and $Q_\tau/P_\tau = \bar{\phi}$ all $\tau \leq t$. Then given $\theta_{it} = 0$ and agent i holds x units of money, $\bar{f}_{2it}$ specifies to propose $(\xi(z(x)), z(x))$, where $z(x) = x$ if $x \leq 1$ and $z(x) = 1$ if $x > 1$; given $\theta_{it} = 1$, $\bar{f}_{2it}$ specifies to accept an offer (y, z) iff $y \leq \xi(z)$.

(b) The history (h^{t-1}_1, S_t) has $S_\tau = \{s\}$ all $\tau \leq t$ and when $t > 1$, $Q_\tau/P_\tau = \bar{\phi}$ all $\tau \leq t-1$. Then $\bar{f}^{post}_{1it}$ specifies an order (q_{it}, p_{it}) which under the price $\bar{\phi}$ adjusts agent i's post-trading money holdings to unity.

(c) When $t = 1$, $\bar{f}^{allc}_{1it}$ specifies to surrender s.

(d) When $t > 1$, the history h^{t-1}_1 has $S_\tau = \{s\}$ and $Q_\tau/P_\tau = \bar{\phi}$ all $\tau \leq t-1$. If the history h_{2it-1} has o_{it-1} in the form of (y, z) with $y \leq \xi(z)$ for some z and $\varsigma_{it-1} = 1$, then $\bar{f}^{allc}_{1it}$ specifies to surrender s; otherwise, $\bar{f}^{allc}_{1it}$ specifies to surrender 0.

(e) The history has either $S_\tau \neq \{s\}$ or $Q_\tau/P_\tau \neq \bar{\phi}$ for $\tau \leq t$. Then in period t, whenever agent i acts, $\bar{f}_{it}$ specifies the same action as f°_i, the strategy of agent i in the purely monetary equilibrium f°.

To verify that $\bar{f}$ is an equilibrium, there are five points worth of checking and all pertain to the equilibrium path. Let $v(\bar{y}) = U(\bar{y})/(1-\beta)$ and $v(y^\circ) = U(y^\circ)/(1-\beta)$.

1. A buyer carrying $x \in (0, 1)$ units of money does not gain by deviating to offer a seller to trade $z \leq x$ units of money with some $y > \xi(z)$ units of goods. The buyer's no-deviation payoff v_1 is $u(\xi(z)) + \beta(x - z - 1)\bar{\phi} + \beta v(\bar{y})$; his deviation payoff v_0 is $\beta s + \beta\phi^\circ(x - 1) + \beta v(y^\circ)$. By (8), $u(\xi(z)) \geq \beta\bar{\phi}z$ so $v_1 \geq \beta\bar{\phi}(x - 1) + \beta v(\bar{y})$. Then by (5), $v_1 \geq v_0$.

2. A seller carrying $x \geq 0$ units of money does not gain by deviating to reject an offer $(\xi(z), z)$. The seller's no-deviation payoff v_1 is $-\xi(z) + \beta\bar{\phi}z + \beta\bar{\phi}(x-1) + \beta v(\bar{y})$; his deviation payoff v_0 is $\beta s + \beta\phi^\circ(x-1) + \beta v(y^\circ)$. By (8), $\xi(z) \leq \beta\bar{\phi}z$. Then by (5), $v_1 \geq v_0$.

3. A seller carrying $x \geq 0$ units of money does not gain by deviating to not reject an offer (y, z) with $y > \xi(z)$ and $z < 1$. The seller's no-deviation payoff v_1 is $\beta s + \beta\phi^\circ(x-1) + \beta v(y^\circ)$; his deviation payoff v_0 is $-y + \beta\phi^\circ z + \beta s + \beta\phi^\circ(x-1) + \beta v(y^\circ)$. Without loss of generality, let $\tilde{z} < 1$. If $z \leq \tilde{z}$, then $y > \xi(z)$ and $\xi(z) = \beta\phi^\circ z$ imply $y > \beta\phi^\circ z$; if $z > \tilde{z}$, then $y > \xi(z)$ and $\beta\phi^\circ z < u^{-1}(\beta\bar{\phi}z) = \xi(z)$ (the last inequality uses (9)) imply $y > \beta\phi^\circ z$. So $v_1 \geq v_0$.

4. An agent does not gain by deviating to surrender $s' \neq s$ to the reallocating system. It suffices to consider $s' = 0$ and an agent who does not hold any money at the start of the period. The agent's no deviation payoff v_1 is $-\bar{\phi} + v(\bar{y})$; his deviation payoff v_0 is $-\phi^\circ + s + v(y^\circ)$. By (5), $v_1 \geq v_0$.

5. The trading post is cleared at the price $\bar{\phi}$, that is, $1 = \arg\max_{x \geq 0} \nu(x)$, where $\nu(x) = 0.5\varphi(x) + (0.5\beta - 1)\bar{\phi}x$ and $\varphi(x) = \max_{0 \leq z \leq x} u(\xi(z)) + \beta\bar{\phi}(x - z)$. Without loss of generality, let $\tilde{z} < 1$. By (8), $\varphi(x) = u(\beta\phi^\circ x)$ for $x \in [0, \tilde{z}]$, $\varphi(x) = \beta\bar{\phi}x$ for $x \in (\tilde{z}, 1)$, and $\varphi(1) = \bar{y}$. So $\nu(x) = 0.5u(\beta\phi^\circ x) + (0.5\beta - 1)\bar{\phi}x$ for $x \in [0, \tilde{z}]$, $\nu(x) = \beta\bar{\phi}x - \bar{\phi}x < 0$ for $x \in (\tilde{z}, 1)$, and $\nu(1) = 0.5u(\bar{y}) + (0.5\beta - 1)\bar{\phi}$. Using (5), $\bar{y} > \beta\bar{\phi}$, and $y^\circ = \beta\phi^\circ$, we have $\nu(1) > 0.5u(\beta\phi^\circ) + (0.5\beta - 1)\phi^\circ \equiv \nu^\circ$. Because $\nu^\circ = \max_{x' \geq 0} 0.5u(\beta\phi^\circ x') + (0.5\beta - 1)\phi^\circ x'$, $\nu(1) > \nu^\circ$ and $\bar{\phi} > \phi^\circ$ imply $\nu(1) > \nu(x)$ for $x \in [0, 1)$. Notice that $x \mapsto \nu(x)$ is strictly concave over $[1, \infty)$, and the right derivative of ν at 1 is $\nu'_{-}(1) = [0.5\beta u'(\bar{y}) + (0.5\beta - 1)]\bar{\phi}$. By $u'(y^\circ) = 2/\beta - 1$ and $\bar{y} > y^\circ$, $\nu'_{-}(1) < 0$ so $\nu(1) > \nu(x)$ for $x > 1$. This completes the proof.

7 Concluding remarks

Starting from a series of seminal works of Kiyotaki and Wright [10, 11, 12], the main ingredient of monetary theory is bilateral interaction between a pair of persons with specialized tastes and production skills, the key of the long lasting narrative that money is useful in a society as it overcomes difficulty of single coincidence of wants (i.e., one person can serve another but not vice versa). The old narrative, however, is insufficient for essentiality of money from a modern perspective. Indeed, the repeated-game literature has a series of results on how cooperation may be sustained with imperfect monitoring. Imperfect monitoring in settings with single coincidence of wants naturally takes the form that one person's service to another cannot be directly observed by a third party. The repeated-game literature shows that this sort of imperfect monitoring need not rule out cooperation absent money and, in particular, contagious equilibria may work (see Araujo [3] for an early contribution). Monetary theory responds by fine-tuning models to eliminate contagious equilibria in specific and maintain essentiality of money in general (see Kocherlakota [13] and Wallace [17, 18] for general discussion on essentiality of money).

Remarkably, contagious equilibria sustain cooperation by collective punishment while money sustains cooperation by individual rewards. The takeaway message of this paper is that in the presence of tradable physical objects which can offer some individual rewards exogenously, the optimal institution may be a mixture of endogenous individual rewards and collective punishment. Some readers may wonder what may be conditions for the optimal institution to be completely monetary. This goes beyond to the scope of the present paper and is left for the future research.

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