Debasements and Small Coins: An Untold Story of Commodity Money

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Abstract

This paper draws quantitative implications for certain historical coinage issues by adapting a fiat-money multiple-denomination model. The model is parameterized to match some key monetary characteristics in late medieval England. A small coin has a more prominent role than small change; thus a shortage of small coins is very costly for poor people and when commerce advances, it is very costly for all people. A debasement may effectively supply substitutes to small coins in shortage but, when small coins are produced from precious metals, debasements cannot be an ultimate solution because precious metals are practically indivisible. So producing money from precious metals may have a far greater cost untold by the literature—commercial advancement would inevitably confer a prominent role on small coins but indivisibility of precious metals restrains technical availability of those coins.

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1 Introduction

Debasements of coins were not rare in medieval Europe: when a type of coin was debased, i.e., the precious-metal content in the coin was reduced, a person could take bullion or old coins of the type to a mint in exchange for new coins. Often debasements were implemented following the public complaints about inconvenience caused by shortages of small coins; see Sargent and Velde [20]. Such complaints were widely recorded, leading to an influential view that the small-coin provision is a big problem for commodity money; see Cipolla [3], Redish [14], and Sargent and Velde [20]. The complained inconvenience ought to mean that shortages of small coins were costly for people, which may be odd to a modern person (a person living in the Eurozone today would not be bothered much by the absence of one-cent or five-cent coins even when he solely uses cash for daily transactions). So, how costly would shortages of small coins be in history? If shortages of small coins were really costly, may debasements alleviate shortages? To what extent did producing money by commodity contribute to the small-coin provision problem? And, is there any new lesson to learn from producing money by commodity in history? To address these issues, we adapt the fiat-money multiple-denomination model of Lee et al. [8] for commodity money. Lee et al. [8] emphasize two factors for agents to hold different denominations: the need for change-giving and the burden for people to carry a bulk of monetary objects. For these two ingredients, imagine a person in the Eurozone insisting on holding one denomination in his wallet: choosing the 500-euro note, he would encounter a change-giving problem; choosing the one-cent coin, he has to carry a huge bag of cash even for grocery shopping.[1]

In our basic model, agents first visit the mint to adjust their portfolios in each period. A portfolio consists of silver coins which do not yield direct utility and a non-coin silver object called jewelry which yields direct utility to agents. Next agents

[1] In popular writings on the currency denomination, a "bulky wallet" is often used to describe the burden for a person to carry many coins or notes. In a recent study, Chen et al. [2] find from a survey by Bank of Canada that shoppers tend to choose payment methods to reduce the number of monetary objects after transactions. The need for change-giving arises when one would face a wide range of transaction values. The early partial-equilibrium denomination models start from a given distribution of transaction values and call a denomination structure optimal if it minimizes some statistic of the number of monetary objects to facilitate transactions; see Telser [26] and Van Hove and Heyndels [28]. Building on the familiar matching model of Trejos and Wright [27] and Shi [22], Lee et al. [8] depart from the partial-equilibrium analysis by endogenizing the distribution of transaction values.

are randomly matched in pairs. In a pairwise meeting, the seller produces goods in exchange for coins with the buyer. The model is parameterized to match some key monetary characteristics of late medieval England. During this period, per capita holdings of silver in money varied but 35 grams may be a useful reference; pennies were the most used coins while the coinage structure gradually expanded with addition of coins larger and smaller than the penny; silver per penny declined over time but 1 gram is a good reference.

We find that adding coins always improves average welfare but large and small coins contribute by much different ways. Large coins, i.e., coins larger than the penny, are beneficial in reducing carrying costs (it is more costly for the buyer to carry 12 pennies than one shilling even though one shilling consists of the same amount of silver as 12 pennies). Small coins, i.e., coins smaller than the penny, are beneficial because they permit an agent to smooth his consumption by way of spreading his purchasing power previously contained in a penny and used in one transaction into a few more smaller coins; that is, small coins have a role more prominent than small change, conforming well to that small coins were quite valuable in late medieval England.[2] As such, shortages of small coins are very costly at least for poor agents; they are so for all agents if the frequency of monetary transactions increases, a trend in the commodity-money era. In the basic model, agents transact mostly with the smallest coins when they trade frequently. Extending the basic model to allow the buyer and seller to transact a bundle of goods over a span of time which may be terminated stochastically, we find that small coins remain valuable but agents often transact with large coins even when they trade frequently.

The basic model also provides a natural link between debasements and shortages of small coins: debasing the penny alleviates shortages of small coins as it effectively supplies small coins. Explicitly including debasements, we find that there are sizable minting responses to debasements and new pennies cocirculate with old pennies *by weight* (weights of coins are public information). Those patterns are at least qualitatively consistent with historical patterns documented by Rolnick et al. [17].[3] But,

[2] In 1490s, a whole pig would cost 33 pennies and one penny could buy 3.73 kg of salt, 3.56 kg of wheat, 1.20 kg of cheese, or 4.35 kg of wool; see Farmer [4, Tables 4, 7]. These numbers would not be surprising if we match per capita silver with per capita M0, which is about GBP 1,200 today in the U.K.

[3] It is not our position that all debasements in history were carried out because monarchs were benevolent. Some debasements clearly had financial motives; see Spufford [24]. But a debasement can successfully meet the monarch's financial need only if it can attract people to the mint. In this

as good as they may appear, debasements are not an ultimate solution to the small-coin problem if small coins are produced from precious metals. A precious metal is *practically indivisible*, i.e., physical properties of the metal imposes a practical lower bound on the metal content in a coin—a low-fineness coin is easy to counterfeit and a high-fineness but low-content coin is too small to carry; see Redish [14]. In fact, a coin like the farthing (a quarter of penny) is largely impractical; the weight of a high-fineness farthing is around 0.4 grams while the weight of a modern U.S. cent is 2.5 grams.

We are not the first to relate debasements with the small-coin provision in the commodity-money era. In their influential monograph*, The Big Problem of Small Change*, Sargent and Velde [20] build a cash-in-advance model in which debasements are socially beneficial as they alleviate shortages in small coins as in our model. A shortage in the Sargent-Velde model is a demand-side problem—agents economize on holding pennies (small coins)—and, debasing the penny alleviates the shortage because new pennies are *assumed* to cocirculate with old pennies *by tale*, enhancing the incentive to hold new pennies. By contrast, a shortage in our model is a supply-side problem and debasing the penny is a supply-side solution to the problem.

Producing money by commodity is conventionally regarded as a commitment device to prevent over-issuance of money in history. How much would the commitment device cost? There is always an opportunity cost (silver can produce jewelry if not used to produce coins); see Sargent and Wallace [21] and Velde and Weber [29]. In addition, the mint's operation cost is not negligible in Sargent and Velde [20] (the zero minting fees eliminate all non-steady-state equilibria and, hence, resolve the small-coin problem). In our model, minting is costless but there is a far greater cost untold by the literature; that is, commercial advancement would inevitably confer a prominent role on coins like farthings but indivisibility of precious metals restrained the technical availability of those coins.[4]

In the literature, there are a few papers which apply matching models of money to draw quantitative implications of indivisibility of precious metal; see, Kim and Lee [6], Lee and Wallace [7], and Redish and Weber [15, 16]. Compared with these

regard, our model provides an explanation for why monarchs were capable of extracting seigniorage through debasements when people were able to distinguish old coins from new coins.

[4]While it is not our point to argue that matching models are superior to cash-in-advance models in modeling money, it seems fair to say that we are able to say things differently than Sargent and Velde [20] because our model is better suited to accommodate indivisibility of a precious metal.

papers, our paper calibrates the coinage structure more closely to the relevant history and reveals the far greater cost due to indivisibility of precious metals by exploring the meeting-frequency dimension.

The rest of the paper is organized as follows. We spell out the basic model in section 2 and summarize basics for quantitative analysis in section 3. Findings from the basic model and from the two extensions are presented in sections 4 and 5, respectively. We discuss the key finding from our model and the related literature in section 6. Section 7 concludes.

2 The model

This section contains the physical environment (including the monetary institution) of the model, the definition equilibrium, and the intuition for the main findings that are presented in subsequent sections.

2.1 Physical environment

Time is discrete, dated as $t \geq 0$. There is a unit measure of infinitely lived agents. At period t, first each agent knows his type at the period—he becomes a buyer or a seller with equal chance. Next, agents visit a mint that produces monetary items, referred to as coins, from a durable commodity, called silver. Silver has a fixed stock M; it can also be costlessly converted into and back from a product, called jewelry. There are K types of coins; per unit coin k contains $m_k > 0$ units of silver, $1 \leq k \leq K$. We refer to $c(m) \equiv (m_1, ..., m_K)$ as a coinage structure. A unit of jewelry contains m_0 units of silver. Agents choose their portfolios of wealth in silver at the mint by the way described below. There is an exogenous upper bound B on each agent's silver wealth. Following their visit to the mint, agents carry coins (but not jewelry) into a decentralized market where each buyer is randomly matched with a seller. In each pairwise meeting, the seller can produce a perishable good that can only be consumed by the buyer. Trading histories are private information, ruling out credits between the two agents. In the meeting, each agent's wealth portfolio is observed by his meeting partner;[5] the buyer makes a take-it-or-leave-it offer.

[5] This assumption is made by many applications of matching models of money, even when some assets in the portfolios are not liquid and not carried into the meeting (e.g., bonds, capital). Without this assumption, each agent's gain-from-trade becomes private information, greatly complicating

Let $Y_t = \prod_{k=0}^{K}\{0, 1, ..., B/m_k\}$ so $y = (y_0, ..., y_K) \in Y_t$ represents an agent's generic portfolio of wealth in silver at period t, meaning that the agent holds y_0 units of jewelry and y_k units of coin k, $k \geq 1$. Coins may exist at the start of period 0; that is, $m_0\pi_0(y_0, 0, ..., 0)$ may be less than M, where π_0 is the distribution of wealth portfolios in silver among agents at the start of period 0. If the agent visits the mint with $y \in Y_t$, he can choose a portfolio from the set

$$\Gamma_t(y) = \{y' \in Y_t : \sum_{k=0}^{K} m_k(y'_k - y_k) = 0\}. \tag{1}$$

If the agent ends with y' after minting and if he consumes $q_b \geq 0$ (when he is a buyer) and produces $q_s \geq 0$ (when he is a seller) in the decentralized market, then his realized utility at period t is

$$u(q_b) - q_s + v(m_0 y'_0) - \gamma \sum_{k=1}^{K} y'_k. \tag{2}$$

The utility function u (for goods) and v (for jewelry) satisfy $u', v' > 0$, $u'' < 0, v'' \leq 0$, $v(0) = u(0) = 0$, and $u'(0) = \infty$; here we follow Velde and Weber [29] to let silver yield direct utility only when it is held for non-monetary use, which permits us to examine how the stock of money would depend on the coinage structure $c(m)$.[6] In (2), $\gamma > 0$ is the disutility to carry per unit of coin to the decentralized market. The carrying cost is a parsimonious term: it covers the physical cost of moving coins, the time cost of counting coins in making payments, the mental cost of keeping coins from losing, etc. Different components of the per-unit carrying cost may vary with the size of coin differently: the physical cost may be proportional to the size, the time cost may be invariant to the size, and the mental cost may be even inversely proportional to the size (according to Redish [14], one important reason that the size of coin cannot be too small is that such a coin is easily lost). Here we follow Lee et al. [8] to let all denominations have the same per unit carrying cost.[7] Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$.

A key statistic in our analysis is the cost due to shortages of coins. The cost is

determination of terms-of-trade in the meeting.

[6] An alternative approach to commodity money is to not separate monetary and nonmonetary uses; see, e.g., Wallace and Zhu [33].

[7] A more general formulation is that the per-unit carrying cost is coin-specific while the total carrying cost is increasing in the number of coins; such a formulation would not affect our results. Also, we follow Lee et al. [8] to assume that the carrying cost only applies to the trip toward the decentralized market; no result would hinge on this assumption.

measured by adding some coins into the present structure. Say, the coin $m_{K'}$ with $K' = K + 1$ is added to the present structure $c(m) = (m_1, m_2, ..., m_K)$. The cost of the complete shortage of $m_{K'}$ may be measured by the difference in average welfare; average welfare for a coinage structure is the average expected lifetime utility for the steady state (as defined below) associated with the coinage structure. But the steady-state comparison may not be sufficient to tell how inconvenient it was for an individual person in history when he complained about shortages of some coins. For such a person, the complaint was likely based on comparing his real experience from transacting with available coins to a hypothetical scenario where coins in shortages were all available to him. The real experience corresponds to what an agent in our model may feel under the steady state associated with $c(m)$ given his current wealth status y; the hypothetical scenario corresponds to what the agent may feel in the equilibrium (as defined below) following a sudden switch of the coinage structure from $c(m)$ to the one with $m_{K'}$ being added.

We formulate the sudden change in the coinage structure as an unanticipated shock to the coinage structure. In the basic model, a shock is referred to as a *structure* shock; that is, it adds some types of coins into the pre-shock coinage structure (section 5 introduces another sort of shock to the coinage structure, referred to as a *debasement* shock). In the post-shock economy, the set of individual portfolios Y_t and the set of portfolios feasible from minting $\Gamma_t(y)$ for $t \geq 0$ are defined by the same way as in the pre-shock economy for a $c(m)$ distinct from the pre-shock economy. Silver contents in all coins are public information in the pre-shock and post-shock economies all the time.

2.2 Equilibrium

We describe equilibrium conditions by a same set of constructs for the pre-shock and post-shock economies, with the understanding that the suitable Y_t and $\Gamma_t(y)$ are applied. For each period t, the set of constructs consists of three probability measures on Y_t, denoted π_t, θ_t^b, and θ_t^s, and three value functions on Y_t, denoted w_t, h_t^b, and h_t^s. Here $\pi_t(y)$ is the fraction of and $w_t(y)$ is the value for agents holding the wealth portfolio y before agents know their period-t types; and $\theta_t^a(y)$ is the fraction of and $h_t^a(y)$ is the value for buyers (sellers, resp.) holding y right after visiting the mint at t when $a = b$ ($a = s$, resp.). As defined below (see (8) and (9)), $h_t^a(y)$ does not include the period payoff from jewelry. Thus, in terms of h_t^a, the portfolio-choice problem for an agent holding y at the mint is expressed as

$$g_t(y, h_t^a) = \max_{y' \in \Gamma_t(y)} h_t^a(y') + v(m_0 y_0'), \; a \in \{b, s\}; \tag{3}$$

that is, the agent chooses a portfolio in the mint to maximize the sum of the post-minting value $h_t^a(y')$ and the period payoff $v(m_0 y_0')$ from jewelry. In terms of w_{t+1}, the trade in a pairwise meeting between a buyer with y_b and a seller with y_s solves the maximization problem

$$f_t(y_b, y_s) = \max_{(q,l)} u(q) + \beta w_{t+1}(y_b - l) \tag{4}$$

subject to

$$-q + \beta w_{t+1}(y_s + l) \geq \beta w_{t+1}(y_s) \tag{5}$$

and $l \in L(y_b, y_s)$, where

$$\begin{aligned} L(y_b, y_s) = \{ & l \in Y_t : l = l_b - l_s, l_b, l_s \in Y_t, l_{b,0} = l_{s,0}, \\ & \text{and } \forall k \geq 1, \; l_{b,k} \leq y_{b,k}, l_{s,k} \leq y_{s,k} \} \end{aligned} \tag{6}$$

is the set of feasible coin transfers between the buyer and the seller. Here, l is the vector that represents the payment (in different coins) made by the buyer. The constraint (5) says that the buyer has all the bargaining power. As it is optimal for the buyer to offer (q, l) such that the seller's participation constraint in (5) is binding, one may infer the purchasing power of i units of coin k as $\beta[w_{t+1}(y_s + i m_k) - w(y_s)]$: the purchasing power of one unit of coin k therefore is equal to the purchasing power of $m_k/m_{k'}$ units of coin k' if $m_k/m_{k'}$ is an integer. In (6), $l_{b,0} = l_{s,0}$ says that jewelry cannot be used for the payment; $l_{b,k} \leq y_{b,k}$ says that the buyer's transfer of coin k (as part of his payment to the seller) cannot exceed his holdings of coin k; and $l_{s,k} \leq y_{s,k}$ says that the seller's transfer of coin k (as part of her change-giving to the buyer) cannot exceed her holdings of coin k.

Given h_t^b and h_t^s, the function w_t satisfies

$$w_t(y) = 0.5 g(y, h_t^b) + 0.5 g(y, h_t^s). \tag{7}$$

As implied by the maximization problem in (4), the function h_t^s satisfies

$$h_t^s(y) = \beta w_{t+1}(y) - \gamma \sum_{k=1}^{K} y_k. \tag{8}$$

Given w_{t+1} and θ_t^s, the function h_t^b satisfies

$$h_t^b(y) = \sum_{y'} \theta_t^s(y') f_t(y, y') - \gamma \sum_{k=1}^{K} y_k. \tag{9}$$

Given π_t, the measure θ_t^a satisfies

$$\theta_t^a(y') = \sum_{y} \pi_t(y) \lambda_1^a(y'; y), \;\; a \in \{b, s\}, \tag{10}$$

for some $\lambda_1^a(.; y) \in \Lambda_1[y, h_t^a]$, where $\Lambda_1[y, h_t^a]$ is the set of measures that represent all randomizations over the optimal portfolios for the maximization problem in (3). Given θ_t^b and θ_t^s, the measure π_{t+1} satisfies

$$\pi_{t+1}(y) = \sum_{(y_b, y_s)} \theta_t^b(y_b) \theta_t^s(y_s) [\lambda_2(y; y_b, y_s) + \lambda_2(y_b - y + y_s; y_b, y_s)] \tag{11}$$

for some $\lambda_2(.; y_b, y_s) \in \Lambda_2[y_b, y_s, w_{t+1}]$, where $\Lambda_2[y_b, y_s, w_{t+1}]$ is the set of measures that represent all randomizations over the optimal transfers of coins for the maximization problem in (4) and $\lambda_2(y; y_b, y_s)$ is the proportion of buyers with y_b who leave with y after meeting sellers with y_s.

Definition 1 *In each of the pre-shock and post-shock economies, a monetary equilibrium is a sequence $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ that satisfies (3)-(11) all t and $\sum_{\{y \in Y_t : y_k = 0, k \geq 1\}} m_0[\theta_t^b(y) + \theta_t^s(y)] < 2M$ some t for a given π_0 and for the applicable Y_t and $\Gamma_t(y)$; a monetary steady state is a tuple $(w, \theta^b, \theta^s, \pi)$ such that $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ with $(w_t, \theta_t^b, \theta_t^s, \pi_t) = (w, \theta^b, \theta^s, \pi)$ all t is a monetary equilibrium.*

For existence, we maintain a simple sufficient condition

$$\frac{B - \underline{m} - 0.5M}{B - \underline{m}} u \left[\frac{\beta(v(B) - v(B - \underline{m}))}{1 - \beta} \right] > v(B) + \frac{\beta}{1 - \beta} [v(B) - v(B - \underline{m})] + \gamma, \tag{12}$$

where $\underline{m} \equiv \min_{k \geq 1} m_k$ is the silver content in the smallest coin in $c(m)$. The condition in (12) says that $\underline{m}$ is not too close to the upper bound B on silver wealth and that compared with some utility from consuming produced goods (the u term), the cost γ of carrying a coin is not too great, and the utility from jewelry (the v terms) is much limited; notice that there is no monetary equilibrium if $\underline{m} = B$, γ is sufficiently great, or the jewelry utility is sufficiently large.

Proposition 1 *In each of the pre-shock and post-shock economies, there exists a monetary equilibrium for a given π_0 and there exists a monetary steady state.*

Proof. See the appendix. ■

2.3 Intuition

Our model is a straightforward adaption of the fiat-money model of Lee et al. [8] for commodity money. As noted above, the model of Lee et al. [8] and earlier partial-equilibrium models emphasize change-giving and the carrying cost as the necessary ingredients to induce people to hold different denominations. What we are going to demonstrate is that when different denominations are demanded, some denominations can be quantitatively far more important than others for welfare and, in particular, small coins are the more important denominations in the commodity-money era because their role goes beyond change-giving.

The following is the relevant intuition. The basic problem for an agent in our model, as the basic problem of agents in Bewley models and as in other models with decentralized markets (e.g., Molico [12], Menzio et al. [11], and Jin and Zhu [5]) is an intertemporal-allocation problem. In our model, as in other denomination-structure models, monetary objects must be indivisible; otherwise, change-giving is not a concern at all and agents only need the largest coin—they can divide the coin for payments when necessary.

Indivisibility can have strong influence on the intertemporal allocation. To simplify the matter, let money be fiat money and let us leave aside the carrying cost. Moreover, let us abstract away dependence of a buyer's spending on the wealth of his potential meeting partner and imagine that each buyer can obtain λx amount of goods by spending x amount of (nominal) wealth for some constant λ. Denote by $x(z)$ the optimal current spending of a buyer when he starts the date with wealth z. Fix $z > 0$; define the sequence $\kappa(z) = \{x_n\}$ by setting $z_1 = z$, $x_n = x(z_n)$, and $z_{n+1} = z_n - x_n$. Then we have $\upsilon(z_n) = 0.5[u(\lambda x(z_n)) + \beta\upsilon(z_{n+1})] + 0.5\beta\upsilon(z_n)$, where υ is the value function defined on the start-of-date wealth before one knows his type, implying

$$\upsilon(z_n) = \frac{0.5}{1-0.5\beta}u(\lambda x_n) + \frac{0.5}{1-0.5\beta}\beta\upsilon(z_{n+1}). \tag{13}$$

(The value function w in section 2.2 is defined on the set of portfolios; because minting is costless, one may equivalently define the value function on the set of wealth levels.) By (13) and the definition of $\kappa(z)$,

$$\upsilon(z) = \sum_{n=1}^{\infty}(\frac{0.5}{1-0.5\beta})^n u(\lambda x_n). \tag{14}$$

By (14), an agent with wealth z can obtain the expected discounted utility $\upsilon(z)$ if he does not produce anything when he becomes a seller and if he spends x_n at the nth time when he becomes a buyer; that is, the sequence $\kappa(z)$ of spendings is *an* optimal solution to the agent's intertemporal allocation problem.[8]

To proceed, let us first use divisible money as the reference. With divisibility, the optimality of $\kappa(z)$ implies the familiar intertemporal consumption-smoothing condition

$$u'(\lambda x_n) = (\frac{0.5}{1-0.5\beta})u'(\lambda x_{n+1}). \tag{15}$$

For CRRA functions, (15) leads to $x_{n+1} = \phi(\frac{0.5}{1-0.5\beta})x_n$, where the function ϕ satisfies $\phi' > 0$ and $\phi(1) = 1$. It follows that $x_1 = z/L$, where $L = \sum_n [\phi(\frac{0.5}{1-0.5\beta})]^n$, which determines the divisible-money optimal sequence $\kappa(z)$ of spendings.

Now let monetary objects be indivisible and let ξ denote the size of the smallest denomination. Suppose ξ, z, and L are all rational numbers. Given N, there exists ξ_N such that if $\xi \leq \xi_N$ then the agent with z can follow the divisible-money optimal sequence $\kappa(z)$ of spending for at least N times. Thus, when N is sufficiently large and $\xi \leq \xi_N$, indivisibility has much limited influence on the agent's expected lifetime utility thanks to discounting. (Needless to say, this is a partial-equilibrium argument as it treats the divisible-money value function as the indivisible-money value function. Zhu [34] establishes a general-equilibrium result that when the smallest denomination can be arbitrarily small, the allocation with indivisible money can be arbitrarily close to the allocation with divisible money.)

But what if ξ is not small, say, $\xi/z > 1/L$ (implying $\xi > x_1$)? Then the spending that best approximates the divisible-money sequence $\kappa(z)$ is $x_n = \xi$ for $n \leq z/\xi$, i.e., the agent only spends ξ when he is a buyer. Thus, indivisibility may have substantial influence—the agent may consume too much each time (with respect to the divisible-money case) and use up money too quickly so that consumption becomes far more unsmooth than the divisible-money counterpart.

A few remarks are in order. First, so far we consider the influence of indivisibility on an agent with an arbitrary wealth level z. The average wealth level is a suitable wealth level to examine the overall influence of indivisibility on the economy; that

[8] There is an alternative optimal solution which requires the agent to produce as a seller; the agent actually acts according to this alternative solution in equilibrium (otherwise when he is a seller, the buyer he meets cannot consume). There are two optimal solutions because in each pairwise meeting, the buyer takes all surplus-from-trade, leaving the seller indifferent between producing and not producing. For our purpose, the solution $\kappa(z)$ is convenient to illustrate how indivisibility affects the agent's intertemporal allocation.

is, the ratio of ξ to the average stock of money M measures the overall degree of indivisibility. Second, indivisibility is not a problem for fiat money. Indeed, whatever ξ is, fiat money has the freedom in choosing M so that ξ/M can be arbitrarily small. Third, indivisibility is a problem for the historical commodity-money system because it encounters both a constraint in M and a constraint in ξ; the latter constraint stems from the practical lower bound on the metal content per coin. Fourth, paying by lotteries can make the influence of indivisibility less substantial. Lotteries, however, would not be a feasible solution for the medieval people. Fifth, even when indivisibility has substantial influence, large coins are demanded by agents as long as carrying coins is costly.

Last but not least, the size m_1 of the smallest denomination matters more if the utility function u is more concave or the meeting frequency meeting frequency F is higher. Indeed, given m_1, with more curvature in the utility function, there is a larger loss in welfare due to the constraint imposed by indivisibility on consumption smoothing (on the flip side, a reduction in m_1, a debasement, has a greater gain in welfare). Also, notice that one period in the model is the span of the calendar time that covers one round of trade, and a higher trading frequency means that n periods correspond to a shorter span of the calendar time; the annual discount factor being fixed, given m_1, running out of money after n periods (the best approximation of indivisible coins to the divisible-money consumption smoothing) becomes more costly for the agent (on the flip side, a reduction in m_1 has a greater gain in welfare).[9]

Ultimately, how substantial the influence of indivisibility may be is a quantitative issue. As such, our analysis below is quantitative.

3 Basics for quantitative analysis

This section summarizes basics for our quantitative analysis.

[9]A higher F under a constant β is quantitatively equivalent to a higher β under a constant F. When F is fixed, a higher value of β leads to a higher value of $\frac{0.5}{1-0.5\beta}$ in (15) and, hence, implies a lower value of x_1 in the divisible-money $\kappa(z)$. A lower value of divisible-money x_1 means that before running out of money, the difference between the indivisible-money spending and the divisible-money spending becomes larger; this enlarged difference enlarges the welfare cost. By this reasoning, if we introduce persistence to the idiosyncratic shock (i.e., if one is a buyer today, then the probability to be a buyer tomorrow is greater than 0.5), then the welfare loss shall be enlarged (as 0.5 in $\frac{0.5}{1-0.5\beta}$ is replaced with a larger number).

3.1 Parameterization

The most important parameters in our analysis are the average silver wealth M and the coinage structure $c(m)$. We choose those parameters to approximate relevant monetary characteristics of the late medieval England. During this period, England gradually expanded its coinage structure by adding silver coins larger than the penny (d), namely, the halfgroat (2d), groat (4d), and shilling (12d), and by adding coins smaller than the penny, namely, the halfpenny (1/2d) and farthing (1/4d). In our analysis, we mainly work with coinage structures that represent this expansion. Specifically, we expand the *single-coin structure* with the penny in one direction by adding the halfgroat, groat, and shilling sequentially and in another direction by adding the halfpenny and farthing. Together with the *complete structure* that consists of all those coins, we have seven representative structures: (1) (single), $(2, 1)$, $(4, 2, 1)$, $(12, 4, 2, 1)$, $(1, 1/2)$, $(1, 1/2, 1/4)$, and $(12, 4, 2, 1, 1/2, 1/4)$ (complete).

The silver content per penny declined over time in history but 1 gram is a good reference. One unit of silver in the model corresponds to 1 gram. So in a coinage structure $c(m)$, 1/2 represents the halfpenny, 1 represents the penny, 2 represents the halfgroat, and so on. We set the silver content in jewelry m_0 at 60; we have in mind that a regular tablespoon weighs around 60 grams in making this choice. We set the average silver holdings M at 35. With our choices of m_0 and M, agents turn out to hold most of the silver in coins so the per capita silver in money falls in the mid of the estimated range for England in the fifteenth century (see Allen [1, p. 607]). We set $B = 3M$ in our analysis. This upper bound on wealth in silver is not restrictive in that it is reached by a negligible measure of agents (the measure is bounded above by 10^{-13} in steady states for our chosen parameters).

We set the annual discount factor at 0.9, $u(x) = x^{1-\sigma}/(1 - \sigma)$ with $\sigma = 0.5$, and $v(x) = \varepsilon x/F$. The value of the annual discount factor, forms of u and v, and the value of σ are consistent with those in the related studies based on similar models (e.g., Lee and Wallace [7] and Redish and Weber [15]). The low annual discount factor has little influence on our main results and, in particular, as is clear below, a higher value of the annual discount factor would only imply a higher welfare cost of shortages of small coins. When people have F rounds of pairwise meetings per year, the discount factor is $\beta = 0.9^{1/F}$. We use $F = 12$ as the baseline value but it should be noted that many results presented below use higher values of F.

The literal interpretation of jewelry is luxury goods. While there is no obvious reference for the marginal utility of luxury goods, it seems reasonable to be conservative by choosing a small value. Indeed, in our model jewelry covers all silver in non-monetary use and, in history much of silver in this use was hoarded. But if ε is too small, the stock of money moves little following a shock to the coinage structure. We set $\varepsilon = 0.01$. With this value, one unit of silver in jewelry yields a utility equivalent to 0.04% of the steady-state per capita consumption per round and we can observe dependence of the stock of money on the coinage structure. As ε, there is no obvious reference for γ. But smaller values for γ seem more preferable than larger; apparently, one exerts tiny effort to carry a coin. We guide our choice of γ by examining values that are sufficiently close to zero (in both absolute and consumption-equivalent terms) and generate sufficient post-shock minting responses. We present our results at $\gamma = 10^{-5}$. This value is equivalent to 0.001% of the steady-state per capita consumption per round.

We do the robustness check. The main patterns of the presented results hold when σ varies from 0.5 to 1.5, ε varies from 0.001 to 0.05, γ varies from 10^{-4} to 10^{-6}, and when v has some strict curvature. On a general level, the intuition in section 2.3 may help us see how a change in a real parameter affects our main result that small coins may be the more important denominations. First, increasing σ adds more curvature to the utility function, which, as noted in section 2.3, leads to more loss in welfare. Second, increasing ε reduces the amount of silver for the monetary use and, hence, makes the shortage of the small coins a more severe problem. Lastly, increasing γ increases the demand for the large coins but shall only marginally affect the welfare loss due to the shortage of the small coins. These are all consistent with what we find in the robustness check. To save the space, we only report part of the check for σ in Appendix C.[10]

3.2 Computational procedure

We compute steady states for different coinage structures and transitional equilibria following unanticipated shocks. Proposition 1 does not tell uniqueness of the monetary steady state in either the pre-shock or post-shock economy. By experimenting over a variety of initial conditions, we find that our algorithm that solves a steady state (see Appendix B.1) always converges to the same steady state for each $c(m)$

[10] The robustness check also covers some coin-specific specification of the carrying cost (see footnote 7). Unreported check results are available subject to request.

given other parameter values (of course, our numerical experiment is not a formal proof). As such, we refer to this steady state as *the steady state* for $c(m)$.

Given the steady state $(w, \theta^b, \theta^s, \pi)$ for $c(m)$, we compute a monetary equilibrium $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ following a shock on $c(m)$ that starts with $\pi_0 = \pi$ and approaches the post-shock monetary steady state. In our algorithm that solves such an equilibrium (see Appendix B.2), we approximate $\lim_{t\to\infty}(w_t, \theta_t^b, \theta_t^s, \pi_t) = (w', \theta^{b\prime}, \theta^{s\prime}, \pi')$ by letting $w_T = w$ for a sufficient large T. We cannot prove that $(w', \theta^{b\prime}, \theta^{s\prime}, \pi')$ is locally stable;[11] even if it is, we cannot prove that there exists $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ with the desired limit property. In fact, for some parameter values outside the aforementioned ranges, our algorithm cannot converge if π_0 is far away from π'.

3.3 Statistics

Let $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ be an equilibrium $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ given a coinage structure $c(m)$. Let $(q_t(y_b, y_s)$,$l_t(y_b, y_s))$ be the optimal solution to the problem in (4) that determines the trading outcome between the buyer with y_b and the seller with y_s; for this meeting, the *net payment* is

$$d_t(y_b, y_s) = \sum_{k=1}^{K} l_{k,t}(y_b, y_s) m_k, \tag{16}$$

and we define the *price* as

$$p_t(y_b, y_s) = d_t(y_b, y_s)/q_t(y_b, y_s). \tag{17}$$

At period t, we define *average meeting output* as

$$Q_t = \sum_{(y_b, y_s)} \theta_t^b(y_b)\theta_t^s(y_s) q_t(y_b, y_s), \tag{18}$$

the average meeting payment as

$$D_t = \sum_{(y_b, y_s)} \theta_t^b(y_b)\theta_t^s(y_s) d_t(y_b, y_s), \tag{19}$$

the average meeting price as

$$P_t = \sum_{(y_b, y_s)} \theta_t^b(y_b)\theta_t^s(y_s) p_t(y_b, y_s), \tag{20}$$

[11]Local stability of $(w', \theta^{b\prime}, \theta^{s\prime}, \pi')$ means that starting from any initial distribution π_0 in a neighborhood of π', there is an equilibrium $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}$ with $\lim_{t\to\infty}(w_t, \theta_t^b, \theta_t^s, \pi_t) = (w', \theta^{b\prime}, \theta^{s\prime}, \pi')$.

the average circulation volume of coin k in silver as

$$CV_{k,t} = [\sum_{(y_b,y_s)} \theta_t^b(y_b)\theta_t^s(y_s)\,|l_{k,t}(y_b,y_s)|]m_k, \tag{21}$$

and *the stock of coin k in silver* as

$$ST_{k,t} = 0.5[\sum_{y_b} \theta_t^b(y_b)y_{b,k} + \sum_{y_s} \theta_t^s(y_s)y_{b,k}]m_k. \tag{22}$$

Let $x_t^a(y) \in \Gamma_t(y)$ be the optimal solution to the portfolio-choice problem in (3); then the minting volume of coin k in silver of an agent in type $a \in \{b, s\}$ holding y is

$$mv_{t,k}^a(y) = \max\{x_{t,k}^a(y) - y_k, 0\}. \tag{23}$$

At period t, we define *the average minting volume of coin k* in silver as

$$MV_{t,k} = 0.5\sum_y \pi_t(y)[mv_{t,k}^b(y) + mv_{t,k}^s(y)] \tag{24}$$

If the equilibrium is a steady state, then objects in (16)-(24) with the subscription t dropped represent the corresponding steady-state values.

The most important statistics for our study pertain to inconvenience or costs due to shortages in some coins. We use the average expected discounted utility

$$W = \pi \cdot w$$

to measure *average welfare* in the steady state $(w, \theta^b, \theta^s, \pi)$ for a coinage structure $c(m)$. When comparing the steady state $(w, \theta^b, \theta^s, \pi)$ for some $c(m)$ with the steady state $(w', \theta^{b\prime}, \theta^{s\prime}, \pi')$ for another $c(m')$, we use

$$\Delta_w \equiv W'/W - 1 \tag{25}$$

to measure the *change in average welfare*. If the set of coins in $c(m)$ is a subset of the set of coins in $c(m')$, then Δ_w provides a measurement of average inconvenience or costs due to a complete shortage of coins in $c(m')$ but not in $c(m)$. Below we report along Δ_w the consumption equivalent Δ_c, i.e., the average change in consumption that equates average welfare in the two steady states.

The statistic Δ_w may not be sufficient to tell how inconvenient it was for an individual person in history when he complained about shortages of some coins. Following the discussion that motivates the structure shock in section 2.1, the real experience of the person who complained about shortages in history corresponds to what an agent in our model may feel under the steady state $(w, \theta^b, \theta^s, \pi)$ given his current wealth status; and the hypothetical scenario for the complaining person corresponds to what the agent may feel in the equilibrium $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$ following the structure shock that turns $c(m)$ into $c(m')$; and the inconvenience of the complaining person corresponds to the *change in an individual agent's welfare* following a structure shock, i.e.,

$$\delta(z) \equiv v_0(z)/v(z) - 1, \tag{26}$$

where z is the agent's pre-shock wealth in silver and $(v_0(z), v(z)) = (w_0(y), w(y))$ if z is equal to the amount of silver in portfolio y. Complementary to Δ_w, the statistic $\{\delta(z)\}_z$ offers a more detailed picture about welfare costs due to shortages of relevant coins. Different from the distribution of consumption that is used to compute the average steady-state expected discounted utility, any distribution of consumption that is used to compute the individual expected discounted utility in a steady state is nonstationary; as such, it is not obvious how to define a consumption equivalent for $\delta(z)$ and we choose not to provide one.

4 Comparing coinage structures

This section compares the seven representative coinage structures indicated in section 3.1. Section 4.1 provides baseline steady-state statistics. Section 4.2 analyzes different welfare roles of small and large coins. Section 4.3 illustrates the asymmetric effects of the meeting frequency on importance of small and large coins. Section 4.4 uses the statistic $\delta(z)$ (see 26) derived from the transitional equilibrium to illustrate the individual cost caused by shortages of small coins.

4.1 Baseline steady-state statistics

Table 1 reports the steady-state average meeting output Q, payment D, and price P for each of the seven representative coinage structures; Table 2 reports the average circulation volume CV, stock ST, and minting volume MT for each coin in those structures; the meeting frequency F is at the baseline value 12. Table 1 also reports the change Δ_w in average welfare and the consumption equivalent Δ_c with the single-coin structure as the base.

Two main findings stand out in Table 1. First, adding a new coin to an existing structure improves average welfare W. This finding applies to other values of F. A partial-equilibrium rationale for the finding is that adding a new coin at least weakly

$c(m)$	D	P	Q	Δ_w	Δ_c
single	1.040	0.902	1.193	–	–
(2,1)	1.040	0.902	1.194	0.03%	0.03%
(4,2,1)	1.040	0.901	1.194	0.05%	0.04%
(12,4,2,1)	1.040	0.901	1.194	0.06%	0.05%
(1,1/2)	1.040	1.077	0.979	1.39%	1.23%
(1,1/2,1/4)	1.107	1.170	0.958	1.67%	1.47%
complete	1.105	1.168	0.957	1.73%	1.53%

Table 1: Steady-state statistics.

$c(m)$		1/4d	1/2d	1d	2d	4d	12d	Jewel	Total
single	*ST*			34.93				0.071	35
	CV			1.040					1.040
	MV			0.007					0.007
(2,1)	*ST*			1.000	33.93			0.072	35
	CV			0.960	0.080				1.040
	MV			0.250	0.256				0.506
(4,2,1)	*ST*			1.000	1.182	33.75		0.072	35
	CV			0.969	0.071	0.019			1.059
	MV			0.251	0.394	0.354			0.999
(12,4,2,1)	*ST*			1.000	1.182	3.935	28.81	0.072	35
	CV			0.972	0.068	0.013	$2e^{-11}$		1.053
	MV			0.252	0.394	0.453	0.309		1.408
(1,1/2)	*ST*		0.420	34.58				$7e^{-4}$	35
	CV		0.092	0.948					1.040
	MV		0.101	0.102					0.203
(1,1/2,1/4)	*ST*	0.250	0.389	34.36				0.001	35
	CV	0.118	0.063	0.957					1.138
	MV	0.066	0.134	0.140					0.341
complete	*ST*	0.250	0.386	0.894	1.006	4.175	28.29	0.001	35
	CV	0.122	0.110	0.866	0.088	0.014	$7e^{-5}$		1.200
	MV	0.065	0.143	0.332	0.349	0.464	0.366		1.720

Table 2: Circulation, stock and minting volumes.

improves welfare because an individual agent can choose to not hold the new coin if it does not help him. There is no obvious general-equilibrium force that would overturn the rationale and we find no counter example when experimenting with other structures with hypothetical coins.[12] Secondly, coins smaller than the penny improve average welfare W much more than large coins, i.e., coins larger than the penny; the asymmetry between large and small coins applies to D, P and Q. To be sure, both large and small coins are used in transactions after introduced even though pennies dominate in the circulation volume CV and have the fastest velocity measured by CV/ST; see Table 2.

There are a few notable observations from Table 2. First, while the stock of jewelry is not sensitive to addition of large coins, it is to addition of small coins, a sign that adding small coins makes the use of silver as money a more attractive option. Second, jewelry and the largest coin tend to absorb more than 90% of silver not used for the transaction purpose. This proportion does not vary much when we vary m_0 from 60 down to 30 but the split between the largest coin and jewelry may vary substantially (agents choose jewelry as the store of value if they can afford it). Third, a larger minting volume need not imply that the coin is more useful in transactions. Indeed, while large coins contribute to more than 70% of the total minting volume, they facilitate less than 3% of the total transaction values.

To see the spending patterns, we conduct the following simulation under the structure $(12, 4, 2, 1)$ for an agent whose initial wealth w_0 is the average wealth (35d) and who turns into a buyer in all future pairwise meetings. At the first period, the agent carries into the decentralized market all types of coins: two shillings (12d), two groats (4d), one halfgroat (2d) and one penny (1d). Depending on the wealth status of the seller in his pairwise meeting, the agent may spend either one penny (with probability 99.93%) or one halfgroat (with 0.07%). At the second period, if he did not spend the half groat in the first period, then mints the halfgroat into two pennies; otherwise, he mints a groat into two pennies and one halfgroat. He spends one penny then with probability 1. In general, once the agent runs out of pennies, he takes a larger coin

[12] To our best knowledge, the literature has not reported that adding a new denomination reduces welfare in the model of Lee et al. [8]. But supplying a new coin itself can be costly; at least, it is costly to produce and maintain the equipment. This cost is abstracted away in our model. Given this cost, the optimal structure certainly depends on parameters related to the supply of coins; if a denomination is rarely used (think of the imaginary 99-euro note or 3-cent coin), then it shall not be supplied. While interesting, we do not pursue optimality in this line.

current wealth		35	34	33	32	31	...
prob. of spending	a penny (1d)	99.93%	100%	100%	100%	100%	100%
	a halfgroat (2d)	0.07%					

Table 3: Simulated spending sequence of a typical agent with average wealth (35) if he keeps being a buyer in pairwise meetings.

to the mint in exchange for pennies to spend. Table 3 summarizes the above result. Of course, when the agent starts with a larger w_0, he is more likely to spend a coin larger than the penny at first. For example, when $w_0 = 70$ (only less than 0.002% of the population has wealth greater than 70 in equilibrium), at the first period, the agent carries into the decentralized market all types of coins and spends one halfgroat with probability 90.52% and one penny with probability 9.48% (as in the case with $w_0 = 35$, the agent carries shillings and groats but does not spend those large coins); in the subsequent periods, the probability to spend one halfgroat decreases and the probability to spend one penny increases.

4.2 Welfare roles of different coins

To understand how adding a coin may improve average welfare, we first introduce two notions of optimal meeting output. One is *the buyer's optimal consumption* for a meeting in a steady state $(w, \theta^b, \theta^s, \pi)$, output when coins are divisible, i.e., the buyer can transfer any amount of silver contained in coins he carries to the seller (but takes the value function w as given); the corresponding payment is the buyer's optimal payment. Another is *socially meeting optimal output* $q^* \equiv \text{argmax}[u(q) - q]$; observe that an upper bound on

$$(1-\beta)^{-1} \sum \theta^b(y_b)\theta^s(y_s)[u(q(y_b, y_s)) - q(y_b, y_s)] \tag{27}$$

is obtained if $q(y_b, y_s) = q^*$ all (y_b, y_s) and that W in the steady state can be approximated by the term in (27) (W and the term in (27) are not exactly equal because some silver is held as jewelry and there are costs to carrying coins). Let us use $c(m) = (1)$ as the reference structure and use adding the halfpenny as the example to understand the welfare role of a small coin. We proceed by three steps.

Step 1. The upper part of Figure 1 displays distributions of meeting output and payment and distributions of the buyer's optimal consumption and payment for $c(m)$= (1). Clearly, buyers in most meetings intend to have lower output and payment

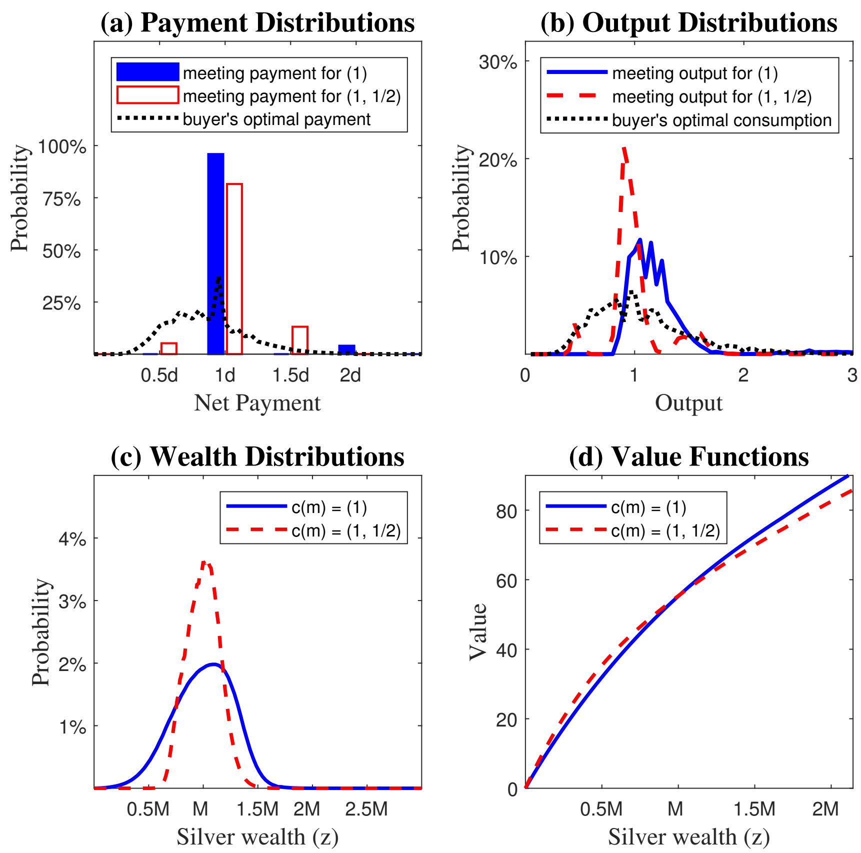

Figure 1: Upper: distributions of payment and output for $c(m) = (1)$ and $(1, 1/2)$, and of buyer's optimal payment and consumption. Lower: wealth distributions and value functions for $c(m) = (1)$ and $(1, 1/2)$.

if they can divide pennies into smaller pieces. With a reduction in the silver content of the smallest coin from 1 to 0.5, the agent benefits provided that the value function (which determines the purchasing power of coins) is unchanged: he can spread his consumption over more periods at a more individually efficient consumption level; this is the *partial-equilibrium* consumption smoothing effect indicated in section 2.3.

Step 2. Now we use the step-1 partial-equilibrium effect to understand the general-equilibrium effects from adding the halfpenny on the steady-state distribution and value function. The lower part of Figure 1 displays the steady-state distributions and value functions. After the halfpenny is added, there is some change in the value function—the value of holding wealth $z < M$ increases and the value of holding $z > M$ decreases. It is anticipated that adding the halfpenny has a greater benefit if

z is smaller. But why does adding the halfpenny actually hurt for a large z? There is a countering effect when the value function tends to move up; that is, the buyer's surplus-from-trade is reduced as the seller's reservation value increases. The net effect is positive (upward) for z below some level but it may be negative for z above another level. It is also anticipated that the distribution should be more concentrated when agents have an option to spend less than one penny (while the magnitude of the concentration is quite striking).

Step 3. Building on the step-2 general-equilibrium effects, there are two distinct perspectives for us to see why adding the halfpenny improves welfare. First, average welfare by definition is the inner product of the distribution and value function. As the value function is largely concave, the concentration of the distribution is the main contributor behind the welfare improvement. Second, the change in the value function may lead the amount of goods purchased by one penny to fall below unity in most meetings; as a result, average welfare improves as output in most meetings moves much closer to q^*—with $c(m) = (1)$, average meeting output Q exceeds $q^* = 1$ by 19% and adding the halfpenny draws Q 3% below q^*. Why does the amount of goods purchased by one penny fall? This is because meeting output is decreasing in the seller's reservation value given the same amount of silver in a payment (for $c(m) = (1, 1/2)$, a majority of transactions still involve a one-penny payment and some output slightly less than unity; see the upper part of Figure 1).

That with $c(m) = (1)$, Q exceeds q^* by 19% is consistent with the finding of Kim and Lee [6]. That with $c(m) = (1, 1/2)$, the halfpenny is not often involved in change-giving may be attributed to our assumption that each agent knows his type when making the portfolio choice in the mint. Although knowing one's type in advance may reduce inefficiency caused by one side of the meeting not being able to give change, measured by ex ante welfare, the benefit may be limited. Why? Think of an agent with the average wealth level in the equilibrium when agents do not know their types in advance. It is always feasible for the agent to choose a portfolio with only the smallest coins so that change-giving will not be a problem. But in equilibrium, the agent does not choose such a portfolio; that is, any ex ante benefit from further reducing the probability of not being able to give change is bounded above by the extra cost from carrying more coins. Because the carrying costs are small in our numerical exercises, it follows that ex ante benefit from reducing the probability of not being able to give change is small. This small welfare-improving effect does not contradict to the significant welfare role of small coins: one result pertains to that the agent intends to spend one halfpenny when the halfpenny is feasible but he chooses to carry pennies; another pertains to that the agent desires to spend one halfpenny but the halfpenny is not feasible.

Apparently, what the addition of the halfpenny can achieve cannot be achieved by the addition of a large coin. Let us keep $c(m) = (1)$ as the reference structure and consider adding the halfgroat. The halfgroat improves average welfare by a much different channel—it saves the carrying costs. With $c(m)=(1)$, on average agents carry 35 coins but on average they only pass one gram of silver per meeting; see Table 2. In other words, most agents carry a lot of silver in pennies not for the transaction purpose; those agents cannot afford jewelry, a more attractive store of value yielding direct utility. With $c(m)=(2,1)$, on average agents still pay one gram of silver per meeting for transactions so on average carrying one penny still meets the transaction purpose but, agents who rely on pennies as the store of value under the single-coin structure can now turn to halfgroats. Indeed, on average agents carry 18 coins (one penny and 17 halfgroats)—the size of their wallets reduces by almost 50%.[13]

4.3 The role of the meeting frequency F

We move on to examine how much the finding demonstrated above would be affected by an increase in the meeting frequency F. We are interested in the increase in F because it captures a fundamental change in the real economy of the relevant part of history—advancement of commerce.

Table 4 displays statistics corresponding to those in Table 1 for $F = 24$. Compared with Table 1, we observe two dramatic changes in Table 4. First, average meeting output Q exceeds q^* by a large margin—more than 100%—absent of small coins. This is not surprising. An increase in F makes per unit of silver contained in coins more valuable—it serves more rounds of transactions for a fixed time frame; so everything else equal, there should be an increase in the purchasing power of a coin. Second,

[13] One may question why agents do not have an option to leave coins serving the store-of-value purpose at home. Our model can accommodate this option. If we further assume that there is no cost of leaving coins at home, then the welfare improvement from adding the halfgroat would become much smaller. Realistically, people would prefer to keeping a smaller number of objects serving as the store-of-value; for this preference, one may compare keeping a jug of cents with keeping a 500-euro note at home. In this regard, our treatment is a simplified one that does not distinguish the cost associated with coins being carried with the cost associated with coins not being carried.

$c(m)$	D	P	Q	Δ_w	Δ_c
single	1.000	0.476	2.142	–	–
(2,1)	1.000	0.476	2.143	0.02%	0.01%
(4,2,1)	1.000	0.476	2.143	0.03%	0.01%
(12,4,2,1)	1.000	0.476	2.143	0.03%	0.02%
(1,1/2)	0.511	0.445	1.174	28.68%	15.55%
(1,1/2,1/4)	0.521	0.532	0.988	30.05%	16.28%
complete	0.521	0.531	0.988	30.13%	16.32%

Table 4: Steady-state statistics under $F = 24$.

consistent with the explanation given at the end of section 2.3, adding small coins has far greater effects: the halfpenny improves average welfare by 28% and reduces both the average payment and output by around 50%.

Let us concentrate on comparing what happens at $F = 24$ with what happens at $F = 12$ after the halfpenny is added to the single-coin structure. When the penny is the only available coin, one penny becomes so valuable at $F = 24$ that it can induce most sellers to produce $q = 2$ and almost all buyers only spend one penny per transaction. So buyers would choose to spend and consume much less if they can divide pennies into small pieces than buyers at $F = 12$. As a result, the distribution becomes more concentrated at $F = 24$ than at $F = 12$ after the halfpenny is added. Of the most importance is that adding the halfpenny has a much stronger up-shifting effect on the value function at $F = 24$ than at $F = 12$. Indeed, the effect is strong enough to dominate the countering effect due to the rise in the seller's reservation value—the entire value function shifts upward after adding the halfpenny. In addition, at $F = 24$, one halfpenny can be sufficiently valuable to induce most sellers to produce q around unity and most buyers to spend one halfpenny per transaction; indeed, the halfpenny dominates in the transaction volume and has the highest velocity when it is available.[14]

What if F rises further? At $F = 48$, average meeting output moves to 2 under $c(m) = (1, 0.5)$. As one may perceive, this renders a great welfare role to the farthing. Moreover, the farthing dominates in the transaction volume and has the highest velocity after it is added to $c(m) = (1, 0.5)$.

[14]If F declines, say, to 1, then the halfgroat dominates in the transaction volume and has the highest velocity after it is added to the single-coin structure.

4.4 Individual cost due to shortages of small coins

Here we turn to the individual statistics $\{\delta(z)\}_z$ to illustrate the individual cost due to shortages of small coins. Because halfpennies and farthings were main targets of public complaints about shortages for the part of history in concern, we use $c(m) = (12, 4, 2, 1)$ as the reference pre-shock structure. We refer to the structure shock that adds the halfpenny as the *halfpenny shock* and the structure shock that adds the halfpenny and farthing as the *halfpenny-farthing shock*. For the purpose of comparison, we also examine the *sixpence shock* that adds a large coin, namely $m_k = 6$, to the reference structure.

The statistic for the sixpence shock is consistent with the finding of the welfare role of large coins based on the steady-state comparison. Specifically, the $\delta(z)$ is positive all z but bounded above by 0.001% at the baseline F, meaning that the lifetime improvement of an agent who benefits the most from the sixpence shock is offset by the costs to carrying 70 coins into the decentralized market once. So even though filling in the gap between the shilling and groat benefits everyone, no one would complain if the gap is left there.

Now we turn to the halfpenny and halfpenny-farthing shocks. Figure 2 displays two sets of statistics $\{\delta(z)\}_z$ in graphs for those two shocks at the baseline F; a graph in the figure is referred to as a $\delta(z)$ curve, with the wealth level z in the horizontal axis and the change in the individual welfare $\delta(z)$ in the vertical axis. The two curves in Figure 2 share two same patterns. First, the change in the individual welfare is decreasing in his pre-shock wealth and the average statistics Δ_w for these two shocks (which can be inferred from Table 2) highly underestimate inconvenience felt by poor people when coins in concern are in shortage.[15] Second, adding small coins makes rich agents worse off. This pattern is consistent with the steady-state comparison in section 4.2 (see the lower part of Figure 1) and can be explained by the effect that counters the consumption-smoothing effect discussed there.

When agents meet more frequently, the first pattern for the baseline F remains and, consistent with the steady-state comparison, the second pattern disappears be-

[15] In our model, agents are all alike ex ante and, hence, an agent's wealth status is transient. But because agents trade in pairs, on average it takes a long time for an agent to transit from his current wealth status to a somewhat different one; for example, for an agent with wealth z =1.2M, it takes on average more than 5000 periods for his wealth level to fall to 0.8M under our chosen parameter values. So while $\delta(z)$ is a snap shot by definition, it would indicate an agent's inconvenience for a much lengthy time frame according to his current status.

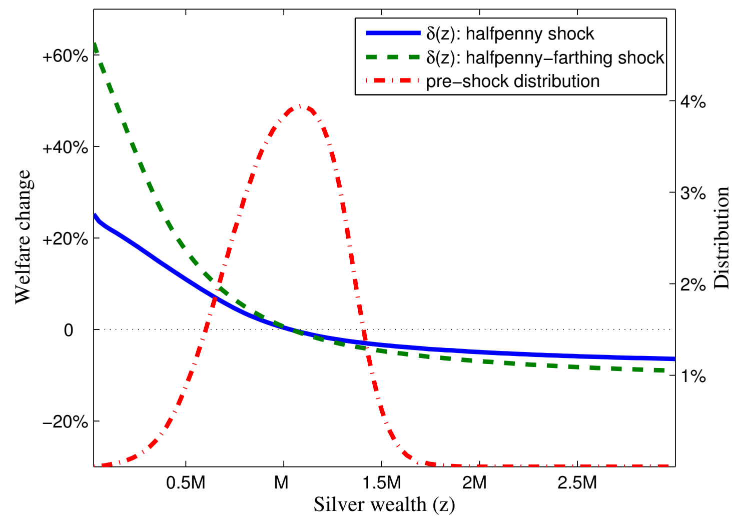

Figure 2: Left axis: changes in individual welfare ($\delta(z)$) under the halfpenny structure shock and under the halfpenny-farthing structure shock. Right axis: pre-shock steady-state distribution.

cause the consumption-smoothing effect is the dominant factor for all agents. The upper part of Figure 3 displays the $\{\delta(z)\}$ curve for the halfpenny shock at $F = 24$; the bottom part of Figure 3 displays the $\{\delta(z)\}$ curve for the halfpenny-farthing shock at $F = 48$. Those curves reveal universal unhappiness for shortages of small coins as the meeting frequency increases. Universal unhappiness prevails even when the farthing is available as long as the trade is sufficiently frequent; for example, if the shock adds $m_k = 1/8$ to the complete structure, then $\delta(z)$ ranges from 66.98% to 19.91% at $F = 120$.

The numbers attached to $F = 120$ for $m_k = 1/8$, together with columns Δ_w and Δ_c in Tables 1 and 4 and Figures 2 and 3, deliver the main finding from the basic model: when people meet more frequently, they strongly demand the smallest coin to be smaller (and poorer people demand more strongly) but their demand may not be satisfied simply because there is a physical lower bound on how small a coin may be.

5 Two extensions

This section presents findings from two extensions of the basic model.

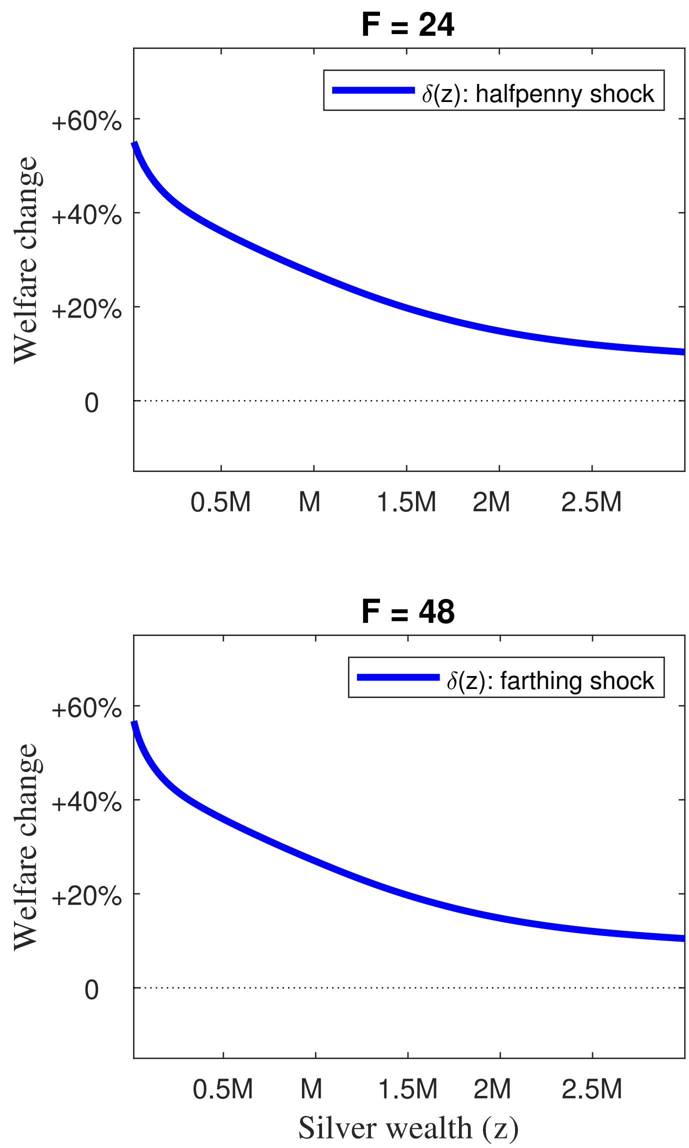

Figure 3: Changes in individual welfare ($\delta(z)$) under the halfpenny structure shock with $F = 24$ (upper); and the alternative farthing structure shock with $F = 48$ (lower).

5.1 Responses to debasements

As documented by Rolnick et al. [17], following a debasement, there is a large increase in the minting volume and new coins cocirculate with old by weight. Rolnick et al. [17] refer to their findings regarding large minting volumes and cocirculation by weight as *the debasement puzzle*—a puzzle based on the presumption that a bundle of old coins are perfect substitute to a bundle of new coins with the same amount of precious metal. This presumption implies that people should not melt old coins in the mint for new coins if there are additional costs stemming from minting, which were minting fees in history.

In our model, two bundles of coins with the same amount of silver need not be perfect substitute. Given great inconvenience due to shortages of small coins illustrated in section 4, it may be anticipated that agents in our model are attracted to the mint to obtain new (lighter) pennies if the penny is debased. To explicitly study how agents respond to debasements, now we let the coinage shock in section 2 be a debasement shock (i.e., we replace a structure shock with a debasement shock).

A debasement shock occurs at period 0 (as a structure shock) and it is represented by a fixed period $\bar{t} > 0$, a set of integers $\{1, ..., J\}$, and a mapping d from J to the set of integers. Here, J is the number of coins of the pre-shock structure being debased, coin $d(j)$ for each $1 \leq j \leq J$ in the pre-shock structure is debased, and $m^o_{d(j)}$ is the amount of silver contained in coin $d(j)$ in the pre-shock structure. The period $\bar{t}$ is the period for coins $d(1), ..., d(J)$ in the pre-shock structure, to exit from circulation; this exit period is a simplifying way to capture that in history, old coins did eventually disappear for a variety of reasons (lost, deteriorated, etc) not considered in our model. If $t < \bar{t}$, then the set of individual portfolios is

$$Y_t = \prod_{k=0}^{K} \{0, 1, ..., B/m_k\} \times \prod_{j=1}^{J} \{0, 1, ..., B/m^o_{d(j)}\} \tag{28}$$

and the set of portfolios feasible from minting is

$$\Gamma_t(y) = \{y' \in Y_t : \sum_{k=0}^{K} m_k(y'_k - y_k) + \sum_{j=d(1)}^{d(J)} m_j(y'_j - y_j) = 0,\ y^{o\prime}_{d(j)} \leq y^o_{d(j)}\}; \tag{29}$$

in (28), a coin with silver content m_k for some $1 \leq k \leq K$ is from the post-shock structure $c(m)$. If $t \geq \bar{t}$, then Y_t and $\Gamma_t(y)$ are defined by the same way as in the pre-shock economy for $c(m)$.

To stay focused, we study the *penny-debasement shock* and *shilling-debasement*

		1/2d	1d	2d	4d	6d	12d	Jewel	Total
Debasing	*ST*	1.355		1.022	3.931		28.69	0.001	35
penny	*CV*	0.972		0.135	0.071		$1e^{-4}$		1.177
(by 50%)	*MV*	0.317		0.334	0.459		0.377		1.487
Debasing	*ST*		1.000	0.204	3.912	29.81		0.072	35
shilling	*CV*		0.972	0.043	0.050	0.075			1.141
(by 50%)	*MV*		0.252	0.203	0.974	1.085			2.514

Table 5: Steady states before and after the debasement shocks.

shock which debase the penny and the shilling by 50%, respectively, from $c(m) = (12, 4, 2, 1)$. Notably, a 50% debasement was not a norm in history. We choose 50% debasements because it is straightforward to compare the lighter penny with the half-penny and the lighter shilling with the sixpence (for lower-degree debasements, the responding patterns presented below are maintained with less increases in minting volumes). According to our formulation, the *penny-debasement shock* and the *half-penny shock* both make the coin containing 0.5 units of silver available from the mint; the coin containing one unit of silver is no longer available from the mint with the former shock and is still available with the latter shock. We present our results with $\bar{t} = 50$.

Table 5 reports the average stocks, circulation volumes, and minting volumes for the two post-shock steady states; the statistics for the pre-shock steady state can be found in Table 2. Following the penny debasement, much of the silver occupied by jewelry is released to coins even though a new penny contains a less amount of silver than an old penny; this is consistent with the movement of the stock of jewelry following addition of small coins in Table 2 and is a sign that holding silver in money is more attractable than holding silver in jewelry following the debasement. Following the shilling debasement, only a tiny amount of silver occupied by jewelry is released; this is consistent with the movement of the stock of jewelry following addition of large coins in Table 2.

The penny debasement resembles the halfpenny shock and the shilling debasement resembles the sixpence shock in welfare effects. When the penny is debased, $\delta(z)$ ranges from 25.21% to -6.44%. When the shilling is debased, $\delta(z)$ is negative but bounded below by -0.008%; the negative (but insignificant) welfare effect may be attributed to the fact that old shillings are a more convenient store of value than new shillings.[16]

Following each shock, we observe cocirculation of old and new coins by weight before old coins exit. Cocirculation by weight is consistent with what Rolnick et al. [17] find; it is natural in our model because the silver content in each sort of coin is public information. Indeed, the value of each coin in the equilibrium definition is a function of the coin's silver content. We have a "by tale" equilibrium if the function is constant in the silver content and a "by weight" equilibrium otherwise. "By tale" equilibria are eliminated by free minting: if a coin with two grams of silver happens to be equally valued as a coin with one gram, people should mint the former coin into two latter coins. But it is worth noting that even when each coin's metal content is public information, circulation by tale following debasement may present if examining coins one-by-one is costly. Abstracted away from our model, this examination cost may not be ignored if a transaction involves many coins; with this cost, circulation by tale (by weight, resp.) can be an equilibrium outcome if the metal difference in new and old coins is sufficiently small (large, resp.).

Because the mint does not supply old coins, one old coin may be more valued than two new coins (i.e., the purchasing power of one old coin may be higher than the purchasing power of two new coins in a meeting); but the premium of one old coin over two new coins (weighted by the distribution of the seller's portfolio) turns out to be rather limited, never exceeding 0.01% along the transitional path.

Circulation of old shillings differs from circulation of old pennies in one aspect. After the shilling is debased, old shillings get more and more circulated because people can only get this convenient store of value from the decentralized-market trade. After the penny is debased, old pennies get less and less circulated because new pennies are good substitutes and more and more old pennies are melted in exchange for new pennies. The different pattern is presented in Figure 4.

Following each shock, there is a sizable increase in the minting volume; see Figure 5. This is again consistent with what Rolnick et al. [17] find. It should not be surprising that the penny debasement induces a significant minting response because it provides coins much demanded in transactions. But a significant minting response

[16] The definition in (26) is not exactly applicable because values of w_0 for two portfolios need not be equal for two portfolios with the same wealth level. But if the two portfolios are in the support of π_0, then the difference in their values of w_0 turns out to be rather limited. So the definition in (26) provides a good approximation if we treat $v_0(z)$ as the average value of w_0 for portfolios with the same z.

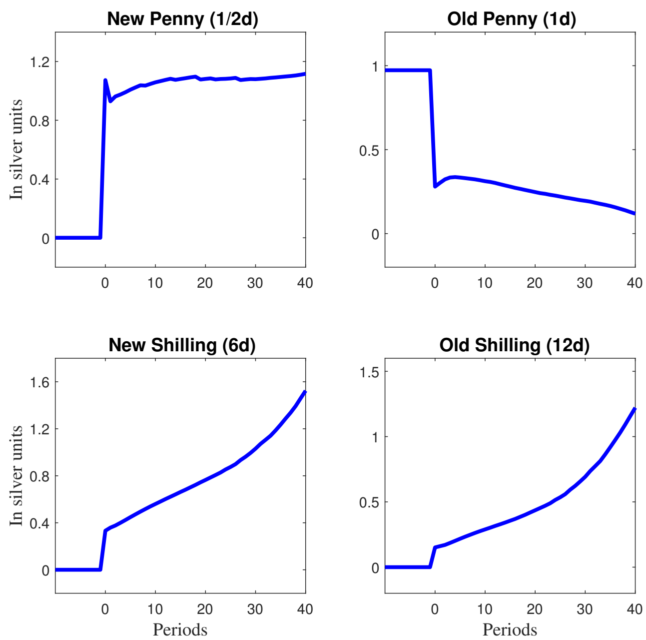

Figure 4: Circulation volumes (CV) of coins following debasement shocks. Upper row: debasing the penny from 1 to 1/2; bottom row: debasing the shilling from 12 to 6.

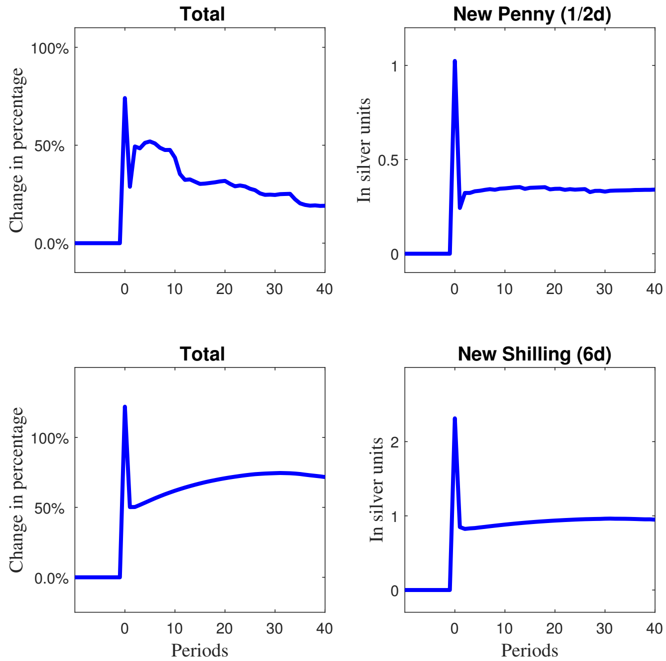

Figure 5: Minting volume (MV) responses following the debasement shocks. Upper row: debasing the penny from 1 to 1/2; bottom row: debasing the shilling with from 12 to 6.

need not imply that a debasement provides coins much demanded in transactions; applied to the shilling debasement, this is analogous to an observation from Table 2: a larger minting volume need not imply that the coin is more useful in transactions.

5.2 High meeting frequency and usage of large coins

The main finding in section 4 is that welfare costs due to shortages of small coins increase as the meeting frequency F increases. When F increases, agents have a strong tendency to use the smallest coins. Little circulation of large coins would cast some doubt on the finding, motivating us to extend the basic model with the buyer-seller interaction in a meeting as follows.

A meeting in period t consists of N stages, indexed by $n \in \{1, ..., N\}$. Each stage consists of three phases. At phase I, if the buyer and seller stay together, each has an option to (endogenously) terminate the meeting. Once the meeting is terminated, the buyer and seller are separated from each other for the rest of period

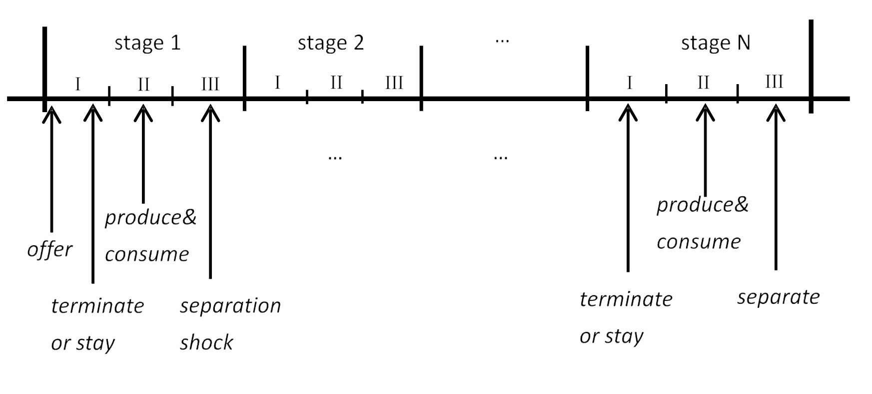

Figure 6: Timeline of the buyer-seller interaction in a given period.

t, particularly implying no production following the termination. At phase II, the seller can produce a good that is consumed by the buyer at the stage provided that the meeting is not terminated. At phase III, there is an i.i.d. separation shock that (exogenously) terminates the meeting with probability ρ_n, where $\rho_n < 1$ if $N > n \geq 1$ and $\rho_n = 1$ if $n = N$. The buyer's utility from consuming the bundle $(c_1, ..., c_n)$, $n \leq N$, is $\sum_{i=1}^{n} u(c_i)$; the seller's disutility from producing the bundle is $\sum_{i=1}^{n} c_i$. One interpretation of this buyer-seller interaction is that a period consists of multiple days, goods in a meeting are time-indexed goods, and a random event may terminate the buyer-seller interaction before the end of the period; e.g., a helper may clean the house for his employer each day in a multiple-day period but the helper may be sick after the first-day house cleaning.[17]

As in the basic model, the buyer makes a take-it-or-leave-it offer. Here an offer is made at phase I of stage 1 before each agent chooses to terminate the meeting or not and it is represented by $(c_1, ..., c_N, l_1, ..., l_N)$, where $c_n \geq 0$ is the seller's production at phase II of stage n conditional on that the meeting has not been terminated at the time and $l_n \in L(y_b, y_s)$ (see (6)) is the buyer's payment conditional on that the meeting is terminated after the seller produces the consumption bundle $(c_1, ..., c_n)$ but before c_{n+1}. If the offer is accepted, then agents move to phase II of stage 1 and

[17] With this interpretation, we may follow Shi [23] to assume that the buyer needs a consumption device to consume and he surrenders the device to the seller at phase n as a collateral if he comes back at phase $n + 1$; in a more complete version, there may be another round of matching in the period among agents who depart from their meetings by the end of phase $n < N$. An alternative interpretation is that a period is short and goods in the meeting are physically distinct; e.g., the helper may clean the house, farm the land, prepare food, etc. for his employer within a day but he may be sick after the land farming.

act according to the offer; otherwise the seller immediately terminates the meeting. Even after the seller accepts an offer, has produced $(c_1, ..., c_{n-1})$, and stays together with the buyer at the start of stage $n > 1$, he can choose to terminate the meeting at phase I of stage n; if he does terminate, then he receives the payment l_{n-1}. Thus if the offer is accepted at stage 1 and neither agent terminates the meeting at stage $n > 1$, then

$$\mu_n = \rho_n \prod_{i=0}^{n-1} (1 - \rho_i) \tag{30}$$

with $\rho_0 = 0$ is the ex-ante probability that the buyer consumes the bundle $(c_1, ..., c_n)$ and pays l_n. Also, $\lambda_{n,n} = \rho_n$ and $\lambda_{n,j} = \rho_j \prod_{i=n}^{j}(1 - \rho_i)$ for $n < j \leq N$ give the probability that the meeting is terminated by the end of stage $n \leq j \leq N$ conditional on that the buyer and seller stay together at the start of stage n and no agent chooses to terminate the meeting at any stage. In terms of w_{t+1}, the buyer's offer solves the optimization problem

$$\max_{(c_1,...,c_N,l_1,...,l_N)} \sum_{n=1}^{N} \mu_n \left[\sum_{i=1}^{n} u(c_i) + \beta w_{t+1}(y_b - l_n) \right] \tag{31}$$

subject to

$$\sum_{j=n}^{N} \lambda_{n,j} \left[-\sum_{i=n}^{j} c_i + \beta w_{t+1}(y_s + l_j) \right] \geq \beta w_{t+1}(y_s + l_{n-1}) \text{ all } n, \tag{32}$$

where $l_n \in L(y_b, y_s)$ and $l_0 = 0$. The constraint (32) implies that the seller does not have incentives to terminate the meeting at phase I of stage n.

The optimization problem (31) can be solved by backward induction. Suppose $(c_1^*, ..., c_N^*, l_1^*, ..., l_N^*)$ is a solution and that after $(c_1^*, ..., c_{n-1}^*)$ has been carried out, the buyer can revise the part of the offer that has not been carried out at phase I of stage n before each agent chooses to terminate the meeting or not. An implication of backward induction is that the buyer is to offer $(c_n^*, ..., c_N^*, l_n^*, ..., l_N^*)$.[18] Hence $(c_1^*, ..., c_N^*, l_1^*, ..., l_N^*)$ is immune to the buyer's redesign at each stage and we may

[18] That is, $(c_n^*, ..., c_N^*, \iota_n^*, ..., \iota_N^*)$ solves the problem

$$\max_{(c_n,...,c_N,l_n,...,l_N)} \sum_{\tau=n}^{N} \mu_\tau \left[\sum_{i=1}^{\tau} u(c_i) + \beta w_{t+1}(y_b - l_\tau) \right]$$

subject to

$$\sum_{j=\tau}^{N} \lambda_{\tau,j} \left[-\sum_{i=\tau}^{j} c_i + \beta w_{t+1}(y_s + l_j) \right] \geq \beta w_{t+1}(y_s + l_{\tau-1}),\ n \leq \tau \leq N,$$

$c(m)$		1d	2d	4d	12d	Jewel	Total
	ST	0.995	1.955	3.826	21.80	6.426	35
Baseline	CV	0.901	1.251	1.004	0		3.156
	MV	0.310	0.571	1.105	0.940		2.926

Table 6: Steady-state statistics of the modified model.

alternatively assume that the buyer makes an offer $(c_n, ..., c_N, l_n, ..., l_N)$ at phase I of each stage n. Because $(c_1^*, ..., c_N^*, l_1^*, ..., l_N^*)$ satisfies the seller's participation constraint at each stage, carrying out the buyer's offer here does not seem to require commitment from the buyer and seller more than commitment for carrying out the buyer's offer in the basic model.

Turning to quantitative properties of this extension, we set $F = 72$, $N = 3$, and $\rho_1 = \rho_2 = 0.1$; that is, people meet once every 5 days and 10% of meetings lasts for 1 day, 9% for 2.5 days, and 81% for 5 days. The steady-state nominal GDP is around 96 pence per year. Table 6 reports usages of coins for $c(m) = (12, 4, 2, 1)$. The shilling is purely a store of value. All other three coins are actively used in transactions; the penny has the highest velocity. Figure 7 displays $\delta(z)$ under the halfpenny shock. The very key behind the patterns in Table 6 and Figure 7 is the following feature in the buyer's equilibrium offer which also applies to other values of N, $\{\rho_n\}$ and F experimented: the payment l_n is roughly proportional to n. Because the ex-ante probability for the buyer to pay l_n is equal to μ_n (see (30)), it follows that independent of the trading frequency, small coins do not dominate in transactions (Table 6) even though small coins have the strong consumption-smoothing effect as in the basic model and shortages of small coins remain very costly (Figure 7).

To further think of the role played by the sequential realization of randomness in the extension, it helps to consider the setting when the number of stages n for each agent to stay together with his meeting partner (if neither chooses to terminate the meeting) is a random variable whose realization is revealed to the two agents at the start of their meeting.[19] Now a buyer's decision has a cutoff-point property: he skips any trading in the present meeting if the realization of n does not reach some cutoff point. Cutoff points are largely determined by the random variable n and the meeting frequency. When the meeting frequency is low, the buyer's spending may vary with

where $l_{n-1} = l_{n-1}^*$.

[19]Such a setting is quantitatively equivalent to the setting with n fixed at one and with a random taste shock to the buyer's utility realized at the start of the meeting.

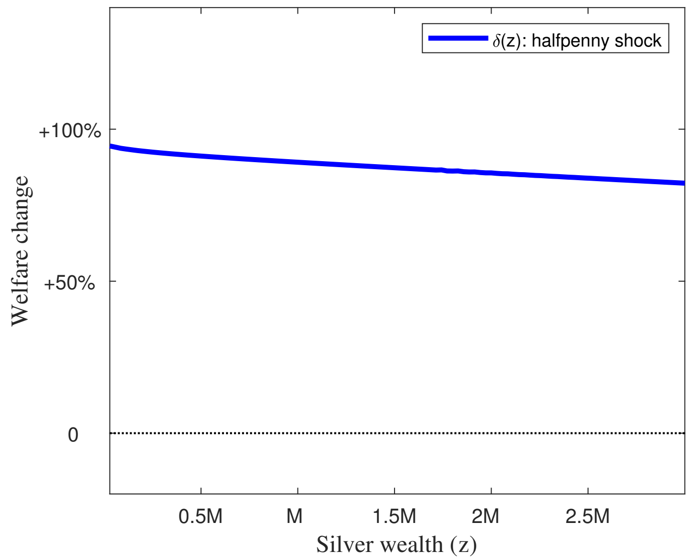

Figure 7: Changes in individual welfare ($\delta(z)$) under the halfpenny structure shock, with $F = 72$, $N = 3$ and $\rho_1 = \rho_2 = 0.9$.

the realization of n after the cutoff point is reached. But when the meeting frequency is high, the buyer only spends one smallest coin in a meeting.

6 Discussion

Here we discuss first our model and next the related literature.

6.1 Our model: the role of M

In section 4, we keep the per capita silver stock M at 35; the main finding is that shortages of small coins are highly costly and the cost goes up dramatically if the meeting frequency increases. How much would the finding be affected if we increase M? Table 7 displays statistics corresponding to those in Table 1 for $F = 12$ (the baseline) and $M = 70$. Comparing the two tables, one may observe the following. First, the quantity theory holds in that the average meeting price more or less doubles as M doubles for each $c(m)$. Second, increasing M moves aggregate output much closer to q^* for $c(m)$ without small coins. Third, the marginal contributions by large coins to average welfare seem not much sensitive to the change in M. Fourth, after M

$c(m)$	D	P	Q	Δ_w	Δ_c
single	1.895	1.866	1.027	–	–
(2,1)	1.894	1.866	1.027	0.02%	0.02%
(4,2,1)	1.894	1.866	1.027	0.03%	0.03%
(12,4,2,1)	1.895	1.866	1.027	0.04%	0.04%
(1,1/2)	1.925	2.005	0.969	0.71%	0.74%
(1,1/2,1/4)	1.987	2.085	0.962	0.72%	0.75%
complete	1.978	2.078	0.961	0.75%	0.78%

Table 7: Steady-state statistics: $F = 12, M = 70$.

increases, small coins remain to make much more significant marginal contributions than large coins but their contributions do decline.

These observations may be understood on the basis that the penny under $M = 70$ is largely equivalent to the halfpenny under $M = 35$; that is, the increase in M is an effective small-coin provision mechanism. If we are free to choose F and M, then we are free to let small coins be welfare significant or not. For example, if we keep F at 12, the marginal contribution to average welfare by small coins is bounded above by 0.003% when M reaches 140; but even if we keep M at 140, average meeting output moves up to 2 and adding the halfpenny has Δ_w above 20% again (as in Table 4) when F reaches 72. Realistically, the increase in F would be the dominant factor in the commodity-money era. There was a rather limited room for per capita silver to grow in the pre-modern time. On the other hand, although transacting a few times per month may approximate a fifteenth-century economy, advancement of commence would inevitably push the meeting frequency far beyond this range.[20] In summary, while the main finding in section 4 is cast in a specific historical period, the basic force underlying the finding would become stronger and, the provision of small coins would become a more serious problem for years ahead of that specific period.

Fiat money provides a perfect solution to the small-coin problem as long as the government can commit to not overissuing fiat money. In our model, fiat money corresponds to the setting that the mint produces coins with a durable material whose supply can be arbitrarily large (silver, then, must all be used for jewelry); fiat money dominates commodity money because of two factors. First, fiat money releases all

[20]The U.S. per capita silver stock was around 800 grams in 1938 and it may not be too restrictive to use 400 grams as an bound for the pre-modern time; see Patterson [13]. This level of silver stock is far from sufficient to bring down the value of small coins when agents transact once per day.

silver to the society for the jewelry service. Second, as noted in section 2.3, fiat money is not subject to any upper bound on the supply of the money-producing material. The second factor is the dominant factor when the meeting frequency is high. These two factors should remain valid when the standard formula is applied, i.e., only small coins become fiat and small token coins can be converted to large silver coins (see Cipolla [3]). To run the standard formula, the mint must hold sufficient reserves in silver. How many reserves are sufficient is an equilibrium outcome. It is left for the future research to explicitly compute a standard-formula model.

6.2 Our model: two missing aspects

Two realistic aspects missing from our model are worth noting. First, we assume minting is free for the purpose to reduce the dimension of the state space. Mints in history charged people to cover the labor and material costs and collect seigniorage. Because it was much more costly to produce farthings than shillings, mints might not produce farthings as demanded given minting fees permitted by monarchs; see, Redish [14, p. 113]. That is, the minting fees would be an important factor for shortages of small coins.

Second, there are no intrinsic heterogeneous characteristics in our model. Strictly speaking, agents in our model only match some class of people in a western European economy in the late middle ages. The economy at that period had the bimetallic system—gold was mostly used in high-value transactions and silver was mostly used in the daily life. The system may be partially contributed by intrinsic heterogeneous characteristics of people (e.g., inherited privileges and skills for more-profitable occupations) that divided people into different classes. A small class of people were rich overall; they would be more involved on market transactions and procure a large proportion of GDP; and they would hold most of gold coins. Agents in our model correspond to people outside this rich class, people who were less involved on market transactions and mainly relied on silver coins for their market transactions.[21]

For people outside the rich class, our model may overestimate welfare significance of small coins because their consumption did not all come from monetary transactions.

[21] This explains our target of the annual nominal GDP per capita at 100 pence for the exercise in Table 6. For the part of history in concern, the annual nominal GDP per capita in England fell in the range from 200 to 400 pence. But 100 pence seems a reasonable target for the non-rich class: if 5 agents in the model count for a household, then the household annually receives monetary incomes around 500 pence, close to the historical data for the non-rich class.

Suppose that monetary transactions only contributed to one third of the consumption and that the consumption of goods and services from monetary transactions entered into the person's utility function as an object distinct from the consumption of goods and services from other means (e.g., barter, credits, and self production); then, one may discount a welfare number by 2/3 to get a more realistic estimate, which is still quite significant. But our model may underestimate welfare significance of small coins for those people because the rich class (excluded from our model) would nonetheless carry a significant stock of silver coins and, hence, make shortages of small silver coins a more severe problem for the non-rich class as a whole.

To include the minting fees, we may assume that each agent incurs some amount of disutility to obtain a unit of coin and that there is an upper bound on the aggregate minting volume for each type of coin; the bound may be exogenous but it can be endogenous and a binding bound describes partial shortages of small coins in history. To accommodate intrinsic heterogeneous characteristics, we may follow Wallace and Zhou [32] by assuming that there are two types of agents who permanently differ in productivity as sellers. Although we do not see a reason for either extension to overturn the main finding, we do anticipate new insights from these extensions. One may examine how a monarch should set minting fees if he intends to maximize seigniorage and how the individual cost due to shortages of small coins depends on his class and his wealth status. Because the state spaces increase dramatically, these extensions are much more challenging to analyze and left for the future work.

6.3 The related literature

In the economic literature, a few papers study shortages of coins or small coins with matching models. Wallace and Zhou [32] study a model with a unit upper bound on money holdings and with some agents less productive than others; they identify a shortage of coins with the concentration of wealth in steady state.

Kim and Lee [6] compare the steady-state aggregate welfare in a model with one sort of coins in fiat money with the steady-state aggregate welfare in a commodity money version of that model; they identify a shortage of small commodity-money coins with a part of the welfare difference contributed by that commodity-money coins are more valuable than fiat-money coins. Lee and Wallace [7] compare the steady-state aggregate welfare in a model with one sort of coins in fiat money by varying the size of the coin; they include the cost of maintaining monetary objects in their analysis; and they conclude that medieval Europe might set the size of the penny right (we suspect that poor agents in their model should get great improvements if the size of the penny is reduced).

Redish and Weber [15] study a model with multiple monetary objects. They build a bimetallic system into the model of Lee et al. [8]: gold and silver coins, respectively, are large and small coins while quantities of gold and silver coins are fixed because gold and silver are distinct metals and because there is no jewelry. Focusing on steady-state comparison, Redish and Weber [15] identify a shortage of a sort of coins with the improvement in the steady-state aggregate welfare when this coin is added; they find that the shortage of small coins can exist. While their model is similar to ours, their parameterization is different. Most importantly, they work with much higher degrees of indivisibility of money and much lower levels of meeting frequency: averages of all coin holdings are no more than 10 and there is one meeting per year when the annual discount factor is 0.9. While a low average of coin holdings tends to strengthen the welfare loss due to shortages of small coins, the low meeting frequency appears powerfully enough to prevent the effect from standing out in their exercises. In their model, there is a probability for a meeting to be a non-trade meeting; varying the trading probability in their model delivers implications qualitatively consistent with varying the meeting frequency in our model, but that variation seems not enough to offset the influence of the low meeting frequency to yield implications quantitatively strong as ours.

Redish and Weber [16] study the essentially same model as we study with two sorts of silver coins. They again focus on steady-state welfare comparison and apply parameters with a low average holdings of coins and a low frequency of meeting. The findings are largely in line with findings in Redish and Weber [15].

As Kim and Lee [6], Lee and Wallace [7], and Redish and Weber [15, 16], we use a matching model to draw quantitative implications of indivisibility of money for the historical commodity-money system. Different from these authors, we calibrate the coinage structure more closely to the relevant history, explore the meeting-frequency dimension, and go beyond steady-state comparison. Going beyond steady-state comparison allows us to quantify an individual's inconvenience due to a shortage of some coins. Our novel contribution lies in (i) demonstrating that the individual inconvenience can be strikingly strong under parameters that do not exaggerate the degree of indivisibility, (ii) illustrating that the poor can experience much more inconvenience than the rich when the poor and rich are defined by their wealth status,[22] (iii) relating the inconvenience to the consumption-smoothing role of small coins and commercial advancement, (iv) showing that the inconvenience can persist even when large coins are substantially used in transactions, and (v) offering an explanation for the debasement puzzle based on the demand for small coins.

Sargent and Velde [20] have ignited the recent interest in small coins in particular and commodity money in general. Their formal analysis is based on the model of Sargent and Velde [19], which replaces cash and credit goods in the cash-in-advance model of Lucas and Stokey [10] with penny and dollar goods—penny goods can only be bought with pennies (small coins) while dollar goods can be bought with dollars (large coins) and pennies. Sargent and Velde identify a shortage of pennies with a binding penny-in-advance constraint, occurring when pennies depreciate relative to dollars; but, as noted by Wallace [31], users of the Lucas-Stokey model usually do not interpret a binding cash-in-advance constraint as a shortage of cash. A reader may observe that jewelry in our model resembles bonds in some cash-in-advance model; that is, jewelry has a higher rate of return than coins but is assumed to be illiquid. This is a valid observation. Nonetheless, different from the Sargent-Velde model, our model does not impose a coin-specific constraint for any sort of coin; moreover, our main finding holds even if jewelry does not yield direct utility (i.e., if money is fiat).

There is a small economic literature that tackles the debasement puzzle. In a cash-in-advance model, Sargent and Smith [18] assume that new and old coins circulate by tale. Under this assumption, agents bring all old coins into the mint in exchange for new coins. On the empirical ground, Rolnick et al. [17] argue that by-tale circulation violates facts documented in the debasement puzzle and that by-tale circulation would have induced a much larger minting volume than observed (data indicate that only a portion of old coins were recoined). In matching models with one unit upper bound on coin holdings, Velde et al. [30] and Li [9] use side payments offered by the mint as incentives for people to bring in old coins in exchange for new coins at a one-to-one rate.

Although minting fees are zero in our model, there are additional costs for an agent to melt old pennies for new pennies—it is costly for the agent to carry more monetary

[22] Relying on steady-state comparison, Redish and Weber [15] define the poor and rich by exogenously-given productivity level to examine the cross-type effects of adding new coins.

objects. In fact, the responding patterns in section 5.1 do not change if agents pay some positive minting fees at the first few periods in the post-shock economy, with minting fees in the disutility form as discussed in section 6.2 (positive fees in the first few periods have limited effects on the dimension of the state space). This exercise is consistent with the finding of Rolnick et al. [17] that there tended to be a period with higher seigniorage *rates* following a debasement.

7 Concluding remarks

Commodity money occupies most part of the monetary history. Compared with the prevailing fiat-money system, the historical commodity-money system is primitive in that its monetary service seems much constrained by the physical properties of precious metals such as scarcity, portability, divisibility, and recognizability. Conventionally thought to be critical, these properties are hard to place in models that many economists are used to and, hence, far from sufficiently explored. Through an off-the-shelf model, our paper demonstrates that the practical indivisibility of precious metals may imply a significant cost for the historical commodity-money system. Plausibly, this cost contributed to the experimentation with a variety of imperfect substitutes to small coins made from precious metals before a commitment device other than precious metals to prevent over-issuance emerged and the final triumph of fiat money after.[23] On the flip side, our paper suggests a reason for one to reconsider commodity money in the presence of the new money-making commodity that is practically divisible and has all other nice physical properties (e.g, bitcoin).

[23] The imperfect substitutes included billon coins, copper coins, pieces cut from coins, foreign coins with less metal content, etc.; see Redish [14, ch 4] for problems with billon coins and copper coins. The standard formula for the small-coin provision prescribes issuing token coins convertible to some precious metal. But convertibility needs commitment. When the state commitment was somehow in place, a society adopted presumably convertible token coins and large-denomination notes. The presumed convertibility finally phased out but the state commitment somehow keeps over-issuance in check.

Appendix

A Proof of Proposition 1

The proof applies the standard fixed point argument. For existence of an equilibrium for a given π_0, it is routine to (i) construct a set S that is compact in the product topology and an element of which is a sequence $\{w_t, \theta_t^b, \theta_t^s, \pi_{t+1}\}_{t=0}^{\infty}$, (ii) construct a mapping F from S to S that is implied by the definition of equilibrium and whose fixed points are equilibria, and (iii) verify that all conditions for the application of Fan's fixed-point theorem are satisfied. So there exists an equilibrium. To show that this equilibrium is a monetary equilibrium, suppose by contradiction the opposite. Without loss of generality, suppose that some agent holds silver wealth B at date 0 and all his wealth is in jewelry. Consider two options of this agent when he is a buyer: minting one unit of the smallest coin and no minting any coin. For the first option, his expected payoff is bounded below by

$$-\gamma + (1 - \frac{0.5M}{B - \underline{m}})u[\frac{\beta\,(v(B) - v(B - \underline{m}))}{1-\beta}] + \frac{\beta}{1-\beta}v(B - \underline{m}).$$

Notice that $1 - 0.5M/(B - \underline{m})$ is a lower bound on the measure of sellers whose wealth levels in silver do not exceed $B - \underline{m}$ and the agent can receive at least $\beta\,(v(B) - v(B - \underline{m}))\,/(1-\beta)$ amount of the good from such a seller. For the second option, his expected payoff is $v(B)/(1-\beta)$. But then (12) implies the first option has a higher payoff, a contradiction. Existence of a monetary steady state can be proof by essentially the same argument.

The proof technique here resembles the one in Taber and Wallace [25] in that the non-monetary equilibrium can be ruled out as a candidate for the fixed point because the money-producing material yields direct utility.

B Numerical algorithms

B.1 Computing a steady state

The algorithm to compute a steady state is essentially an iteration of the mapping that defines the steady state. To begin with, vectorize the $K+1$-state space into a one-dimensional state, and define the value vectors $\{w, g\}$ and distribution vectors

$\{\theta, \pi\}$, $\theta = (\theta^b, \theta^s)$, accordingly. Denote the total possible number of states as S.

1. Begin with an initial guess $\{w^0, h^0, \theta^0, \pi^0\}$, where π^0 and θ^0 are consistent with the total silver stock M.

2. Given post-minting value h^i and pre-minting distribution π^i from i-th iteration, solve the problem (3), and use the solution to update pre-minting value w^{i+1} and post-minting distribution θ^{i+1} .

3. With w^{i+1} and θ^{i+1}, solve the problem as described in (4). Record the terms of trade of each relevant pairs, and update h^{i+1} and π^{i+1} accordingly.

4. Repeat step 2-3 until the convergence criterion is satisfied: $\|w^{i+1} - w^i\| < 10^{-6}, \|h^{i+1} - h^i\| < 10^{-6}$ and $\|\theta^{i+1} - \theta^i\| < 10^{-8}, \|\pi^{i+1} - \pi^i\| < 10^{-8}$.

B.2 Computing a post-shock equilibrium

The computation for a post-shock equilibrium is essentially about iterations on the series of $\Psi \equiv \{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=1}^T$, $h_t = (h_t^b, h_t^s)$ and $\theta_t = (\theta_t^b, \theta_t^s)$, where T is the number of periods it takes for the economy to reach a new steady state. Before computing the transition paths, we first need to compute the post-shock steady state using an algorithm similar to B.1, with the change that options of portfolios containing old coins are unavailable at the mint. Denote this steady state as $\{w_T, h_T, \theta_T, \pi_{T+1}\}$. We also have to translate the distribution from the pre-shock steady state, into the beginning distribution in the debasement environment, denote the beginning distribution as π_1.

1. Take an initial guess $\Psi^0 \equiv \left\{w_t^0, h_t^0, \theta_t^0, \pi_{t+1}^0\right\}_{t=1}^T$, with $w_T^0 = w_T$.

2. Start from the last period T. Given w_T and θ_T^i, solve the pairwise bargaining problem as described in (4), and get h_T^i. Record the implied Markov transition matrix as Λ_T^i. Use h_T^i and π_T^i, solve the problem of minting, and get w_{T-1}^i accordingly. Record the implied Markov transition matrix as Υ_T^i. Then use w_{T-1}^i and θ_{T-1}^i, repeat the previous procedure for problems in period $T-1$. Finally, we will have a new series $\{w_t^i, h_t^i\}_{t=1}^T$. And then use $\{\Lambda_t^i, \Upsilon_t^i\}_{t=1}^T$ and π_1 and generate a new series of distributions $\left\{\pi_t^{i+1}, \theta_t^{i+1}\right\}_{t=1}^T$.

3. Now use $\left\{\pi_t^{i+1}, \theta_t^{i+1}\right\}_{t=1}^T$ and w_T, repeat Step 2 and get $\left\{\pi_t^{i+2}, \theta_t^{i+2}\right\}_{t=1}^T$.

σ	$c(m)$	D	P	Q	Δ_w	Δ_c
0.5	*single*	1.988	2.626	0.766	–	–
	(1,1/2)	1.974	2.654	0.752	0.63%	0.24%
	(1,1/2,1/4)	1.973	2.661	0.749	0.82%	0.32%
1.0	*single*	1.006	1.105	0.977	–	–
	(1,1/2)	0.998	1.268	0.804	3.49%	3.34%
	(1,1/2,1/4)	1.003	1.305	0.780	4.11%	3.94%
1.5	*single*	1.000	0.655	1.697	–	–
	(1,1/2)	0.598	0.782	0.847	24.82%	70.15%
	(1,1/2,1/4)	0.666	0.848	0.800	29.86%	92.26%

Table 8: Steady-state statistics under various σ; $\omega = 0.2$.

4. Repeat 2-3 until the convergence criterion is met: $\max_t \left(\left\|\pi_t^{i+1} - \pi_t^i\right\|\right) < 10^{-8}$, $\max_t \left(\left\|\theta_t^{i+1} - \theta_t^i\right\|\right) < 10^{-8}$, $\max_t \left(\left\|w_t^{i+1} - w_t^i\right\|\right) < 10^{-6}$, and $\max_t \left(\left\|h_t^{i+1} - h_t^i\right\|\right) < 10^{-6}$.

C Robustness check for σ

Here we report part of the robustness check regarding the parameter σ. To accommodate the values of σ no less than unity, we set $u(x) = [(x+\omega)^{1-\sigma} - \omega^{1-\sigma}]/(1-\sigma)$ (for $\sigma = 1$, $u(x) = \ln(x+\omega) - \ln(\omega)$), where $\omega > 0$. The exercise the one in Table 1 with adding smaller coins to single-coin structure under $\sigma = 0.5$, 1, and 1.5, with ω fixed at 0.2. The results are summaries in Table 8 reports; the welfare implications of small coins are consistent with the discussion at the end of section 3.1.

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